QCD Pomeron in a thermostat

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Content

1 Introduction

Hadron-hadron scattering

$$A + B \to A' + B'$$

Mandelstam variables

$$s = (p_A + p_B)^2\,,\; t = (p_A - p_{A'})^2\,,\; u = (p_A - p_{B'})^2$$

Regge kinematics

$$s \gg -t \simeq m^2\,,\; t \leq 0$$

Regge asymptotics

$$M(s,t) = \sum_k \xi_k(t)\, g_k(t)\, \left(\frac{s}{m^2}\right)^{j_k(t)} g_k(t)$$

Regge trajectory and signature factor

$$j_k(t) = j_k^{(0)} + j'_k(0)\, t + ...$$

$$\xi_k(t) = \frac{1 - P_k\, e^{-i\pi j(t)}}{\sin\left(\pi j(t)\right)}\,,\; P_k = \pm 1$$

Regge family

$$S = j_k(M^2)$$

Optical theorem

$$\sigma_t = \frac{Im\, M(s,0)}{s}$$

Pomeron contribution

$$\sigma_t \sim (s/m^2)^{\Delta}\,,\ P_p = 1$$

Pomeron trajectory

$$j_P(t) = 1 + \Delta + j'_P\, t + ...,\ \Delta \ll 1$$

Pomeranchuck theorem

$$\frac{\sigma_t^{pp}}{\sigma_t^{p\bar{p}}} \to 1$$

Pomeron family = glueball states

What will be with these states at the high temperature T?

2 BFKL Pomeron

Leading logarithmic approximation

$$M_{AB}^{A'B'}(s,t) = s \sum_{n=0}^{\infty} (\alpha_s \, \ln s)^n \, a_n(t)$$

Region of applicability

$$\alpha_s \ln s \sim 1, \, \alpha_s = \frac{g^2}{4\pi} \to 0$$

Born amplitude

$$M_{AB}^{A'B'}(s,t)|_{Born} = g \, T^c_{A'A} \, \delta_{\lambda_{A'} \lambda_A} \, \frac{2s}{t} \, g \, T^c_{B'B} \, \delta_{\lambda_{B'} \lambda_B}$$

Gluon reggeization in LLA

$$M_{AB}^{A'B'}(s,t) = M_{AB}^{A'B'}(s,t)|_{Born} \, s^{\omega(t)}$$

Gluon Regge trajectory

$$\omega(-|q|^2) = -\frac{\alpha_c}{4\pi^2} \, N_c \int d^2k \, \frac{|q|^2}{|k|^2 |q-k|^2} \approx -\frac{\alpha_c}{2\pi} \, \ln \frac{|q^2|}{\lambda^2}$$

Gluon production amplitude

$$M_{2\to 1+n} \sim \frac{s_1^{\omega_1}}{|q_1|^2}\, gT^{d_1}_{c_2 c_1} C(q_2,q_1)\, \frac{s_2^{\omega_2}}{|q_2|^2} ... C(q_n,q_{n-1})\, \frac{s_n^{\omega_n}}{|q_n|^2}$$

Reggeon-Reggeon-gluon vertex

$$C(q_2,q_1) = \frac{q_2\, q_1^*}{q_2 - q_1}\,,$$

Impact parameter coordinates

$$\rho_k = x_k + iy_k\,,\ \rho_k^* = x_k - iy_k\,,\ p_k = i\frac{\partial}{\partial \rho_k}\,,\ p_k^* = i\frac{\partial}{\partial \rho_k^*}$$

Balitsky-Fadin-Kuraev-Lipatov equation (1975)

$$E\,\Psi(\vec{\rho}_1,\vec{\rho}_2) = H_{12}\,\Psi(\vec{\rho}_1,\vec{\rho}_2)\,,\ \Delta = -\frac{\alpha_s N_c}{2\pi}\,E$$

BFKL Hamiltonian

$$H_{12} = \ln\,|p_1 p_2|^2 + \frac{1}{p_1 p_2^*}\,\ln\,|\rho_{12}|^2 p_1 p_2^*$$

$$+\frac{1}{p_1^* p_2}\,\ln\,|\rho_{12}|^2 p_1^* p_2 - 4\psi(1)$$

3 Remarkable properties of the BFKL dynamics

1. Möbius invariance (L.L. (1986))

$$\rho_k \to \frac{a\rho_k + b}{c\rho_k + d_k}$$

Eigenvalues $m(1-m)$ and $\tilde{m}(\tilde{m}-1)$ of two Casimir operators with

$$m = \frac{1}{2} + i\nu + \frac{n}{2}\,,\ \tilde{m} = \frac{1}{2} + i\nu - \frac{n}{2}$$

for a principal series of unitary representations.

2. Holomorphic separability of the BFKL Hamiltonian (L.L (1989))

$$H_{12} = h_{12} + h_{12}^{*}\,,$$

$$h_{12} = \ln(p_1 p_2) + \frac{1}{p_1 p_2^{*}}\, \ln\rho_{12}\; p_1 p_2^{*} + \frac{1}{p_1^{*} p_2}\, \ln\rho_{12}\; p_1^{*} p_2 + 2\gamma\,,$$

$$\rho_{12} = \rho_1 - \rho_2$$

3. Holomorphic factorization at large N_c (L.L. (1989))

$$\Psi(\vec{\rho}_1, \vec{\rho}_2, ..., \vec{\rho}_n) = \sum_{r,s} a_{r,s}\, \Psi_r(\rho_1, ..., \rho_n)\, \Psi_s(\rho_1^{*}, ..., \rho_n^{*})$$

$$H = \frac{h + h^{*}}{2}\,, \quad h = \sum_{r=1}^{n} h_{r,r+1}$$

4. Duality symmetry (L.L. (1999))

$$\rho_{r,r+1} \to p_r \to \rho_{r-1,r}$$

5. Integrability (L.L. (1993)). Integrals of motion

$$q_r = \sum_{k_1<k_2<\ldots<k_r} \rho_{k_1k_2}\rho_{k_2k_3}\ldots\rho_{k_{r-1}k_r}\, p_{k_1}p_{k_2}\ldots p_{k_r}$$

6. Equivalence with the integrable Heisenberg model (L.L. (1994) and L.F., G.K.(1995)) with the spins $\vec{M}_k$

$$M_k^z = \rho_k\partial_k\,,\; M_k^+ = \partial_k\,,\; M_k^- = -\rho_k^2\partial_k$$

4 Pomeron wave function

Holomorphic separability

$$H_{12} = h + h^* \,,\; h = \sum_{r=1}^{2} \left[\ln\, p_r + \frac{1}{p_r}\, \ln\left(\rho_{12}\right)\, p_r - \psi(1) \right]$$

Möbius group generators

$$M_3^{(r)} = \rho_r\, \partial_r \,,\;\; M_+^{(r)} = \partial_r \,,\;\; M_-^{(r)} = -\rho_r^2\, \partial_r$$

Casimir operators

$$M^2 = \left(\sum_{r=1}^{2} \vec{M}^{(r)} \right)^2 = \rho_{12}^2\, p_1\, p_2 \,.$$

Eigenvalue equations

$$M^2 f_{m,\widetilde{m}} = m(m-1) f_{m,\widetilde{m}} \,,\; M^{*2} f_{m,\widetilde{m}} = \widetilde{m}(\widetilde{m}-1) f_{m,\widetilde{m}}$$

Conformal weights

$$m = 1/2 + i\nu + n/2 \,,\; \widetilde{m} = 1/2 + i\nu - n/2$$

Pomeron wave functions

$$f_{m,\widetilde{m}}(\overrightarrow{\rho_1},\overrightarrow{\rho_2};\overrightarrow{\rho_0}) = \left(\frac{\rho_{12}}{\rho_{10}\,\rho_{20}}\right)^{m}\left(\frac{\rho_{12}^{*}}{\rho_{10}^{*}\,\rho_{20}^{*}}\right)^{\widetilde{m}}$$

Pomeron energy

$$E_{m,\widetilde{m}} = \varepsilon_m + \varepsilon_{\widetilde{m}} \quad , \quad \varepsilon_m = \psi(m) + \psi(1-m) - 2\psi(1)$$

BFKL Pomeron intercept

$$\Delta = 4\,\frac{\alpha_s}{\pi} N_c\,\ln 2$$

Violation of the Froissart bound

$$\sigma \sim s^{\Delta} > c \ln^2 s$$

5 Pomeron at a finite temperature

Invariants in the t-channel c.m. frame

$$t = 4E^2\,,\; s = -2(\vec{p})^2(1-\cos\theta)$$

Periodicity of fields at a finite temperature

$$\phi(x_4) = \phi(x_4 + \frac{1}{T})$$

Quantization of energies

$$E_l = 2\pi\, i\, l\, T$$

Regge kinematics in the s-channel

$$s >> T^2 \sim -t > 0$$

Cylinder topology and momentum quantization

$$\phi(\vec{\rho}) = \phi(\vec{\rho} + \frac{\vec{e_y}}{T})\,,\; k_y^{(l)} = 2\pi\, l\, T$$

Rescaled variables

$$\rho = x + iy \to \frac{1}{2\pi T}\,\rho \quad , \quad p^{(l)} = \frac{p_x^{(l)} - ip_y^{(l)}}{2} \to \pi\,T\,p$$

Temperature constraints

$$0 < \mathrm{Im}\,\rho < 2\pi \quad , \quad \mathrm{Im}\,p^{(l)} = \frac{l}{2}$$

Gluon Regge trajectory

$$\omega(-\vec{q}^{\,2}) = -\frac{g^2}{8\pi^2}\,N_c\,\Omega(-\vec{q}^{\,2}) \quad , \quad \Omega(-\vec{q}^{\,2}) = \Omega(q)+\Omega(q^*)$$

Holomorphic trajectory

$$\Omega(q) = \frac{\pi T}{2\lambda} + \frac{1}{2}\,\left[\psi(1+iq) + \psi(1-iq) - 2\psi(1)\right]$$

Green function

$$G(\overrightarrow{\rho}_{12}) = G(\rho_{12}) + G(\rho_{12}^*)$$

Holomorphic Green function

$$G(\rho_{12}) = -\frac{\pi\,T}{2\lambda} + \ln\left(2\sinh\frac{\rho_{12}}{2}\right)$$

6 BFKL equation at finite T

Schrödinger equation

$$H_{12}\Psi = \Psi\,,\; H_{12} = h_{12} + h_{12}^*$$

Holomorphic Hamiltonian

$$h_{12} = \sum_{r=1}^{2}\left[\Omega(q_r) + \frac{1}{p_r}\,G(\rho_{12})\,p_r\right]$$

Small-T expansion

$$h_{12} = h_{12}^0 + \sum_{k=1}^{\infty}\frac{B_{2k}}{2k}\sum_{r=1,2}\left[\frac{(-1)^{k+1}}{p_r^{2k}} + \frac{1}{p_r}\,\frac{\rho_{12}^{2k}}{(2k)!}\,p_r\right]$$

Eigenfunction expansion for $Q = 0$

$$\Psi_m(\rho_{12}) = \rho_{12}^m \left[1 - \frac{1}{24} \frac{m(m-1)}{2m+1} \, \rho_{12}^2 + \mathcal{O}(\rho_{12}^4) \right]$$

Integral of motion:

$$A = 4 \sinh^2 \frac{\rho_{12}}{2} \; p_1 \, p_2 \quad , \quad [A, h_{12}] = 0$$

Eigenvalue equation for $t = -4|Q|^2$

$$\left[\frac{Q^2}{4} + \frac{\partial^2}{\partial \rho_{12}^2} \right] \, \Psi(\rho_{12}, Q) = \frac{m(m-1)}{4 \sinh^2 \frac{\rho_{12}}{2}} \, \Psi(\rho_{12}, Q)$$

Eigenfunctions

$$\Psi_1^m(\rho,\, Q) = e^{\frac{i}{2} Q \, \rho} \; (e^{\rho} - 1)^m \; F \left(iQ + m, m; 2m; 1 - e^{\rho} \right)$$

$$\Psi_2(\rho,\, Q) \equiv \Psi_1^{1-m}(\rho,\, Q)$$

Pomeron wave function

$$\Psi_{m,\widetilde{m}}(\vec{\rho}, \vec{Q}) = \sum_{r=(m,1-m)} a_r \, \Psi_r(\rho, Q) \, \Psi_{\widetilde{r}}(\rho^*, Q^*)$$

Periodicity to the shift

$$\rho \to \rho + 2\pi \, i$$

Energy

$$E = \epsilon_m + \epsilon_{\widetilde{m}} \, ,$$

Holomorphic energy

$$\epsilon_m = \psi(m) + \psi(1-m) - 2\psi(1)$$

7 Integrability of the BFKL dynamics

Conformal transformation

$$\rho_r = \ln\, \rho'_r$$

Integral of motion and Hamiltonian in new variables

$$A = -(\rho'_{12})^2 \frac{\partial}{\partial \rho'_1}\, \frac{\partial}{\partial \rho'_2}\,,$$

$$h_{12} = \ln(p'_1\, p'_2) + \frac{1}{p'_1}\, \log(\rho'_{12})\; p'_1 + \frac{1}{p'_2}\, \log(\rho'_{12})\; p'_2 - 2\psi(1)$$

Operator identity

$$\frac{1}{2}\left[\psi\left(1 + z\frac{\partial}{\partial z}\right) + \psi\left(-z\frac{\partial}{\partial z}\right)\right] = \ln\, z + \ln\, \frac{\partial}{\partial z}$$

Pomeron wave function

$$\Psi_{m,\widetilde{m}} = \left(\frac{\frac{1}{2}\sinh\frac{\rho_{12}}{2}}{\sinh\frac{\rho_{10}}{2}\,\sinh\frac{\rho_{20}}{2}}\right)^{m}\left(\frac{\frac{1}{2}\sinh\frac{\rho^*_{12}}{2}}{\sinh\frac{\rho^*_{10}}{2}\,\sinh\frac{\rho^*_{20}}{2}}\right)^{\widetilde{m}}$$

Monodromy matrix for n Reggeons at $N_c \to \infty$

$$t(u) = L_1(u)\, L_2(u)...L_n(u)\,,\ [tr\,(t(u)), h] = 0$$

L-operators

$$L_k = \left(\begin{array}{cc} u + p_k & e^{-\rho_k}\, p_k \\ -e^{\rho_k}\, p_k & u - p_k \end{array} \right)$$

Integrable Heisenberg model with the spins

$$M_k = \partial_k \quad , \quad M_+ = e^{-\rho_k}\, \partial_k \quad , \quad M_- = -e^{\rho_k}\, \partial_k$$

Balitsky-Kovchegov equation for non-zero T

$$\frac{\partial N_{\vec{\rho}_1,\vec{\rho}_2}}{\partial Y} = \bar{\alpha}_s \int \frac{d^2\rho_0}{2\pi}\, \frac{\left|\sinh \frac{\rho_{12}}{2}\right|^2}{4\left|\sinh \frac{\rho_{10}}{2}\right|^2 \left|\sinh \frac{\rho_{20}}{2}\right|^2}$$

$$(N_{\vec{\rho}_1,\vec{\rho}_0} + N_{\vec{\rho}_2,\vec{\rho}_0} - N_{\vec{\rho}_1,\vec{\rho}_2} - N_{\vec{\rho}_1,\vec{\rho}_0}\, N_{\vec{\rho}_2,\vec{\rho}_0})$$

8 Discussions

1. A non-zero temperature in the c.m. system of the t-channel leads to a cylinder type topology of the impact-parameter space in the s-channel.
2. The Regge trajectory of a gluon and its Green function at a finite temperature have the property of the holomorphic separability, which implies a similar property of the BFKL Hamiltonian.
3. The intercept of the BFKL Pomeron does not depend on the temperature.
4. The Pomeron wave function is constructed in an explicit way with the use of the conformal transformation $\rho = \ln \rho'$.
5. At a finite temperature the BFKL Hamiltonian coincides with the local Hamiltonian of an integrable Heisenberg spin model.
6. Is the BFKL dynamics integrable for the case when the impact parameter space is an arbitrary Riemann surface?
7. Is it possible to find a relation between the BFKL and string dynamics?