Reconstruction of spatial resolution of multilayer position-sensitive detectors

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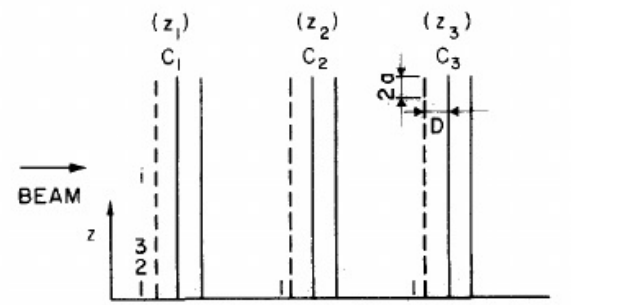
Introduction

- Consider a series of detectors measuring the coordinates of a track.
  Measurement of only one coordinate is considered here.
  No pileup, 100% efficiency $\Rightarrow$ one hit per detecting layer.
  No magnetic field.
  No reference detectors.
  True track coordinates and inclination angles are unknown. $\Rightarrow$
  It is necessary to obtain measurement errors from measurements with unknown errors.

- The assumption of identical layer resolution, only the proportionality coefficient is needed:
  [G. Charpak, et al., NIM 167(1979)455],
  [F. Piuz, et al., NIM 196(1982)451],
  [A. Korytov, et al, NIM A 338(1994)375],
  and many others.



$R_z = \frac{z_1+z_3}{2} - z_2 + \text{misaligment} \Rightarrow$
fwhm of residuals is $2.36\sigma\sqrt{3/2}$ [F. Piuz et al, 1982].

The result is not the resolution of any particular layer,
nor is it an exact average value,
but a value that doesn't have a clearly defined meaning.
This actually prevents

- a more detailed study of the detectors and
- a more accurate analysis of their data.

Introduction (2)

- Estimation of variances "in a linear model" [R. Frühwirth, NIM A243(1986)173].
  Measurement errors are allowed to be unequal.
  Hits in all layers are used for fit, the weights of all hits are unities.
  Details of derivation of the key formula are not clear
  (His formlas "$\mathbf{B} = \mathbf{I} - \mathbf{A}(\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}$", "$\mathbf{B}^{\intercal}\tilde{\mathbf{C}}\mathbf{B} = \tilde{\mathbf{C}}$" does not seem to have a proof, meaning and accuracy of the fitting procedure do not seem to be clear),
  but surely different from my approach.
  The final key formula is similar to a particular case of my result (obtained later):
  $\boldsymbol{\sigma}^2 = (\mathbf{B}^*)^{-1} \cdot \mathrm{diag}(\tilde{\mathbf{C}})$ (here notations of Frühwirth).
  Also a necessary condition for the problem to have a solution
  is claimed by R. Frühwirth: $n \leqslant (n-m)(n-m+1)/2$.
  Here n is the number or layers,
  m be the order of trajectory (2 for straight),
  Four-layer detector: $4 > 2 \times 3/2 = 3 \Rightarrow$ failure :(
  Five-layer detector: $5 < 3 \times 4/2 = 6 \Rightarrow$ success :)
  It may need a proof for non-unity weights and if the equations for covariances are used.

Introduction (3)

- The method of geometric mean: R.K. Carnegie et al, NIM A 538(2005)372, "a *better estimate* of the true resolution" (than inclusive and exclusive) quotes:

Let us assume a track consisting of N measurements with known values y_j, $1 \leqslant j \leqslant N$. The corresponding measured values x_j are distributed around the expected mean $\langle x_j \rangle = a + by_j$ with the standard deviations s_j, where a and b are the track parameters. To determine the resolution of one measurement i it is convenient to choose the coordinate system so that $y_i = 0$. In this case, the residual is given by $\delta_i = a - x_i$, where a can be determined from a least square fit to the track by either including (a^{in}) or excluding (a_i^{ex}) the measurement i. The residual δ_i will be distributed with a standard deviation σ_{δ_i} which is related to s_i, but depends on the coordinates (x_j, y_j) and weights $w_j = 1/s_i^2$ of all measurements.

Expanding, rearranging and collecting terms yields:

$$\sigma^2_{\delta_i^{\text{in}}} = s_i^2 \frac{D_i^{\text{ex}}}{D^{\text{in}}} \qquad \text{(A.7)}$$

and similarly

$$\sigma^2_{\delta_i^{\text{ex}}} = s_i^2 \frac{D^{\text{in}}}{D_i^{\text{ex}}}.$$

The quantities D^{in} and D_i^{ex} are fixed for a given layout and can be calculated to correct the resolution measured, however it is simpler to combine the last two expressions:

$$s_i^2 = \sigma_{\delta_i^{\text{in}}} \sigma_{\delta_i^{\text{ex}}}. \qquad \text{(A.8)}$$

Also [D.C. Arogancia et al, NIM A 602(2009)403]: " the *true* spatial resolution".

MC tests in Ref. [T. Alexopoulos et al., JINST, 9 (2014) P01003]:
"The geometric mean method produces *accurate results when* the test and reference detectors have the same characteristics."

In my opinion, a major problem with this method: weights, which are needed for calculation of redisuals and resolutions, are inverse layer variances,
which are unknown before they are calculated... using themselves.

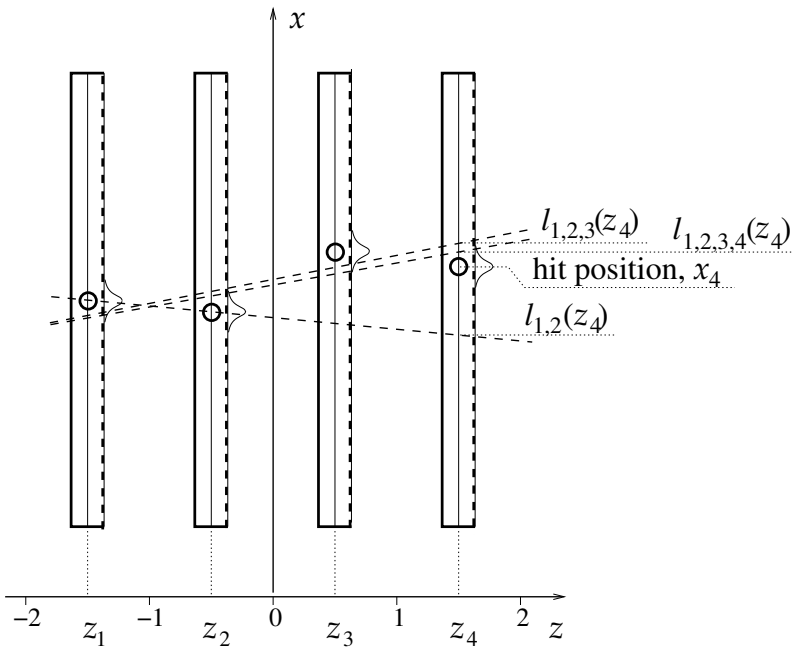
Introduction (4)

> Coming to the layer spatial resolution one has to multiply the residuals for each layer by the correction factors calculated from the diagonal elements of the "Hat matrix" [7]. For ME1/1 CSC layers the correction factors are 1.45 for layers 1 and 6, 1.19 for layers 2 and 5,

- The use of the correction factors "calculated from diagonal elements of "Hat" matrix" [I. A. Golutvin et al., Physics of Part. Nucl. Let., 7(2010) 355].
  The "Hat" matrix is from [A.C. Rencher et al., Linear Models in Statistics, 2008, page 228].
  The resolutions are assumed identical (although can be calculated separately).
  In Rencher notation: $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$, $\boldsymbol{\epsilon} = \sigma^2\mathbf{I}$, $\hat{\boldsymbol{\epsilon}} = \boldsymbol{y} - \boldsymbol{X}\hat{\boldsymbol{\beta}}$, $\boldsymbol{\epsilon}$ is error, but $\hat{\boldsymbol{\epsilon}}$ is residual.
  $\hat{\boldsymbol{\beta}} = (\mathbf{X'X})^{-1}\mathbf{X'y}$, $\hat{\mathbf{y}} = \mathbf{X}\hat{\boldsymbol{\beta}} = \mathbf{X}(\mathbf{X'X})^{-1}\mathbf{X'} = \mathbf{Hy}$, $\hat{\boldsymbol{\epsilon}} = (\mathbf{I} - \mathbf{H})\boldsymbol{y} = (\mathbf{I} - \mathbf{H})\epsilon$.
- Claim of reconstructing 4 resolutions:
  J. Bortfeldt et al., IEEE Trans. Nucl. Sci. 59 (2012) 1252.
  J. Bortfeldt, Springer Theses, 2015;
  Not reproduced.
  Also contradicts to [R. Frühwirth, NIM A243(1986)173] (see slide 4).
- Obtaining resolution from sum of residuals (with assumption of identical layers):
  Unbiased estimator $\Rightarrow$ can be averaged by many tracks.
  $\sigma^2 = \frac{Q^2_{\min}}{N-r}$ [F. James, Statistical Methods..., 2006, Section 8.4.1, page 185],
  [A.C. Rencher et al., Linear Models in Statistics, 2008, page 131] ,
  [Kendall et al. Section 19.9].

Main notations

Example of 4-layer detector
(like Cathode Strip Chamber)



N is the number of layers (in the plot $N = 4$).
z_i are positions of detecting layers.
x_i, $i = 1, 2, 3, 4$ are measured coordinates of hits.
$x_{\mathrm{t},i}$ are true positions of hits.
ϵ_i are errors of measurements: $x_i = x_{\mathrm{t},i} + \epsilon_i$.
$l_{1,2\ldots}(z)$ is the position of the straight line fitted by layers 1, 2... at z.
More formal notation of fitted track: $\hat{l}(\boldsymbol{w}_\nu, \boldsymbol{x}, z_i)$:
Each straight line is fitted with own sets of weights;
$\boldsymbol{w}_\nu$ denotes a vector of all $w_{\nu,i}$ for set of weights ν and for all layer $i \in [1, N]$; some $w_{\nu,i}$ can be zero, which means that the corresponding layers are not used for fit;
$\boldsymbol{x}$ denotes a vector of all x_i.
r_i are residuals $x_i - l_{1,2\ldots}(z_i)$.

$E(\xi)$ is the expectation of any value ξ (which can be x_i, ϵ_i, etc.).
$\sigma(\xi)$ is the standard deviation of any value ξ. $V(\xi) = \sigma(\xi)^2$ is the variance.
$\mathrm{cov}(\xi_1, \xi_2) = E[(\xi_1 - E(\xi_1))(\xi_2 - E(\xi_2))]$ is the covariance.
$\boldsymbol{\xi}$, $E(\boldsymbol{\xi})$, $\sigma(\boldsymbol{\xi})$, $V(\boldsymbol{\xi})$ are vectors with N components.

Notations of expectations, variances and covariances

No systematic shifts and no electric cross talks between layers =>

- the expectation $E(\epsilon_i) = 0$,
- the correlations $\mathrm{corr}(\epsilon_i, \epsilon_j) = 0$,

But $x_{\mathrm{t},i}$ are correlated! => x_i are correlated as well!

Error propagation rules (informal notation of a range of theorems):

- $E\left(\sum_i^N a_i \xi_i\right) = \sum_i^N a_i E(\xi_i)$,
- $V\left(\sum_i^N a_i \xi_i\right) = \sum_i^N a_i^2 V(\xi_i) + 2\sum_i^N \sum_{j=i+1}^N a_i a_j \ \mathrm{cov}(\xi_i, \xi_j)$,
- $\mathrm{cov}\left(\sum_i^N a_i \xi_i, \sum_j^N b_j \xi_j\right) = \sum_i^N a_i \sum_j^N b_j \ \mathrm{cov}(\xi_i, \xi_j)$.

Residuals

The straight line: $l(z_i) = a_1 + a_2 z$.
Find the parameters by minimization of $M = \sum_{i=1}^{N} w_{\nu,i}\big(x_i - l(z_i)\big)^2$,
$w_{\nu,i}$ is the weight that we want to use.
For maximal likelihood this shold be inverse variance, but it is still unknown.
Why not to use different weights if it helps to reconstruct resolutions?..
The optimal line for these weights is denoted by hats:
$\hat{l}(\boldsymbol{w}_\nu, \boldsymbol{x}, z_i) = \hat{a}_1(\boldsymbol{w}_\nu, \boldsymbol{x}) + \hat{a}_2(\boldsymbol{w}_\nu, \boldsymbol{x}) z$.
Denote: $s_\nu = \sum_{i=1}^{N} w_{\nu,i}$, $\overline{x}_\nu = \sum_{i=1}^{N} w_{\nu,i} x_i / s_\nu$, $\overline{z}_\nu = \sum_{i=1}^{N} w_{\nu,i} z_i / s_\nu$,
$\overline{xz}_\nu = \sum_{i=1}^{N} w_{\nu,i} x_i z_i / s_\nu$, $\overline{z^2}_\nu = \sum_{i=1}^{N} w_{\nu,i} z_i^2 / s_\nu$, $D_\nu(z) = \overline{z^2}_\nu - (\overline{z}_\nu)^2$.
Minimization of M results in

$$\hat{a}_2(\boldsymbol{w}_\nu, \boldsymbol{x}) = \frac{\overline{xz}_\nu - \overline{x}_\nu\ \overline{z}_\nu}{D_\nu(z)}\ , \qquad \hat{a}_1(\boldsymbol{w}_\nu, \boldsymbol{x}) = \overline{x}_\nu - a_2 \overline{z}_\nu = \frac{\overline{x}_\nu \overline{z^2}_\nu - \overline{xz}_\nu\ \overline{z}_\nu}{D_\nu(z)}\ .$$

These formulas are not unique, they can be found in many sources in various forms, for example,
[V. K. Grishin et al., Math. Threatment and Interp. of Phys. Exper., 1988, page 115, in Russian]

Residuals for each layer i: $r_i = x_i - \hat{l}(\boldsymbol{w}_\nu, \boldsymbol{x}, z_i)$.
Its variance $V(r_i) = V(x_i - \hat{l}(\boldsymbol{w}_\nu, \boldsymbol{x}, z_i)) = V\big(x_{\mathrm{t},i} + \epsilon_i - \hat{l}(\boldsymbol{w}_\nu, \boldsymbol{x}_\mathrm{t} + \boldsymbol{\epsilon}, z_i)\big)$ depends on known $\boldsymbol{w}$ and unknown $\boldsymbol{\epsilon}$ and unknown $\boldsymbol{x}_\mathrm{t}$ $\Rightarrow$ Application of the error propagation rules is not promising.

Residuals in matrix form

The same in matrix form:
Let $f_j(z)$ be an arbitrary functions, in our case $f_1(z) = 1$, $f_2(z) = z$.
$x_\mathrm{t}(z) = \sum_{j=1}^{k} a_j f_j(z)$, or $\boldsymbol{x}_\mathrm{t} = F\boldsymbol{a}$, where:

$$F = \begin{pmatrix} f_1(z_1) & f_2(z_1) & \dots & f_k(z_1) \\ f_1(z_2) & f_2(z_2) & \dots & f_k(z_2) \\ \dots & \dots & \dots & \dots \\ f_1(z_N) & f_2(z_N) & \dots & f_k(z_N) \end{pmatrix}.$$

Introduce the weight matrix:

$$W_\nu = \begin{pmatrix} w_{\nu,1} & 0 & \dots & 0 \\ 0 & w_{\nu,2} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & w_{\nu,N} \end{pmatrix}.$$

The minimization of

$$M = (\boldsymbol{x} - F\hat{\boldsymbol{a}})^\mathsf{T} W_\nu (\boldsymbol{x} - F\hat{\boldsymbol{a}})$$

resuilts in

$$\hat{\boldsymbol{a}}_\nu = (F^\mathsf{T} W_\nu F)^{-1} F^\mathsf{T} W_\nu \boldsymbol{x}.$$

Then,

$$\boldsymbol{r}_\nu = \boldsymbol{x} - F\hat{\boldsymbol{a}}_\nu.$$

Variances and covariances of residuals and resolution

r_i is a linear combination of components of the vector $\boldsymbol{x}$:
$r_{\nu,i}(\boldsymbol{x}) = x_i - \hat{I}(\boldsymbol{w}_\nu, \boldsymbol{x}, z_i) = \sum_{j=1}^{N} u_{\nu,ij} x_j$, where $u_{\nu,ij}$ are constants that depend on $\boldsymbol{w}$ and $\boldsymbol{z}$, but do not depend on $\boldsymbol{x}$.
The residual calculated with true $\boldsymbol{x}_\mathrm{t}$ should be zero. Therefore, by construction:

$$r_{\nu,i}(\boldsymbol{x}_\mathrm{t}) = x_{\mathrm{t},i} - I(\boldsymbol{w}_\nu, \boldsymbol{x}_\mathrm{t}, z_i) = \sum_{j=1}^{N} u_{\nu,ij} x_{\mathrm{t},j} = 0 \ .$$

Then:

$$r_{\nu,i}(\boldsymbol{x}) = x_i - \hat{I}(\boldsymbol{w}_\nu, \boldsymbol{x}, z_i) = \sum_{j=1}^{N} u_{\nu,ij} x_j =$$

$$\sum_{j=1}^{N} u_{\nu,ij}(x_{\mathrm{t},j} + \epsilon_j) = \sum_{j=1}^{N} u_{\nu,ij} x_{\mathrm{t},j} + \sum_{j=1}^{N} u_{\nu,ij}\epsilon_j = \sum_{j=1}^{N} u_{\nu,ij}\epsilon_j = \epsilon_i - \hat{I}(\boldsymbol{w}_\nu, \boldsymbol{\epsilon}, z_i).$$

The same in matrix notation:

$$\boldsymbol{r}_\nu = \boldsymbol{x} - F\hat{\boldsymbol{a}}_\nu = \boldsymbol{x} - F(F^\mathsf{T} W_\nu F)^{-1} F^\mathsf{T} W_\nu \boldsymbol{x} = (I - F(F^\mathsf{T} W_\nu F)^{-1} F^\mathsf{T} W_\nu)\boldsymbol{x} =$$

$$\boldsymbol{x}_\mathrm{t} + \boldsymbol{\epsilon} - F(F^\mathsf{T} W_\nu F)^{-1} F^\mathsf{T} W_\nu \boldsymbol{x}_\mathrm{t} - F(F^\mathsf{T} W_\nu F)^{-1} F^\mathsf{T} W_\nu \boldsymbol{\epsilon} =$$

$$F\boldsymbol{a} + \boldsymbol{\epsilon} - F(F^\mathsf{T} W_\nu F)^{-1} F^\mathsf{T} W_\nu F\boldsymbol{a} - F(F^\mathsf{T} W_\nu F)^{-1} F^\mathsf{T} W_\nu \boldsymbol{\epsilon} = (I - F(F^\mathsf{T} W_\nu F)^{-1} F^\mathsf{T} W_\nu)\boldsymbol{\epsilon}.$$

Cf. quotation from Rencher, slide 5 (there it is without the weight matrix).

Conclusion: in any residual as a linear function of $\boldsymbol{x}$, we can substitute $\boldsymbol{x}$ by $\boldsymbol{\epsilon}$.

Error propagation

After that we can apply error propagation rules and calualte variances of residuals, taking into account $\mathrm{cov}(\epsilon_i, \epsilon_j) = 0$.

Denote $u^2_{\nu,ij}$ by h_{ij}. Then: $V(r_i) = \sum_{j=1}^{N} h_{ij} V(\epsilon_j)$.

The matrix H consists of elements h_{ij}. Then: $V(\boldsymbol{r}) = HV(\boldsymbol{\epsilon}) \quad \Rightarrow \quad V(\boldsymbol{\epsilon}) = H^{-1}V(\boldsymbol{r})$.

By calculating the cariances from the experimental data, we can obtain the resolutions, provided that the matrix H is not singular.

This is a particular case of covariances of reiduals.

Denote $u_{\nu,ik}u_{\mu,jk}$ by $h^{(\nu,\mu)}_{i,j,k}$. Then: $\mathrm{cov}(r_i, r_j) = \sum_{k=1}^{N} h^{(\nu,\mu)}_{i,j,k} V(\epsilon_k)$.

Now denote by H the $N \times N$ matrix:

$$H = \begin{pmatrix} h^{(\nu_1,\mu_1)}_{i_1,j_1,1} & h^{(\nu_1,\mu_1)}_{i_1,j_1,2} & \dots & h^{(\nu_1,\mu_1)}_{i_1,j_1,N} \\ h^{(\nu_2,\mu_2)}_{i_2,j_2,1} & h^{(\nu_2,\mu_2)}_{i_2,j_2,2} & \dots & h^{(\nu_2,\mu_2)}_{i_2,j_2,N} \\ \vdots & \vdots & \vdots & \vdots \\ h^{(\nu_N,\mu_N)}_{i_M,j_N,1} & h^{(\nu_N,\mu_N)}_{i_N,j_N,2} & \dots & h^{(\nu_N,\mu_N)}_{i_N,j_N,N} \end{pmatrix} .$$

Denote the vector of corresponding covariances $\mathrm{cov}(r_{\nu,i}, r_{\mu,j})$ by $\boldsymbol{y}$.

Then: $\boldsymbol{y} = HV(\boldsymbol{\epsilon}) \quad \Rightarrow \quad V(\boldsymbol{\epsilon}) = H^{-1}\boldsymbol{y}$.

Unfortunately, all possible H-matrices are singular for 3- and 4-layer detectors.

Varying weights does not help, details are later.

Geometric means, inclusive and exclusive residuals

Let $\exists i \quad \Big(w_{\nu,i} > 0 \wedge w_{\mu,i} = 0 \wedge (\forall j \neq i \quad w_{\mu,j} = w_{\nu,j} \geqslant 0) \wedge$

$$(\exists j \neq i \wedge \exists k \neq i \quad (j \neq k \wedge w_{\mu,j} > 0 \wedge w_{\mu,k} > 0))\Big)$$

Meaning: the second set denoted by μ ("inclusive") is identical to the first one denoted by ν ("inclusive") except the zero weight of layer i. There are at least two non-zero weights in the second set.

Choose z such that $z_i = 0$. Let the subscript λ denote either ν or μ when a formula applies to both cases. A non-normalized weighted average of any value v_k by two bars: $\overline{\overline{v}}_\lambda = \sum_k w_{\lambda,i} v_k$. The "ordinary" average: $\overline{v}_\lambda = \overline{\overline{v}}_\lambda / s_\lambda$, $s_\lambda = \sum_k w_{\lambda,k}$. Because $z_i = 0$, we have $\overline{\overline{z}}_\nu = \overline{\overline{z}}_\mu$, $\overline{\overline{z^2}}_\nu = \overline{\overline{z^2}}_\mu$ and $\overline{\overline{xz}}_\nu = \overline{\overline{xz}}_\mu$, so we can omit the indexes ν and μ for these averages.

$$s_\lambda^2 D_\lambda(z) = s_\lambda \overline{\overline{z^2}} - \overline{\overline{z}}^2, \quad s_\nu^2 D_\nu(z) = s_\mu^2 D_\mu(z) + w_\mu \overline{\overline{z}}^2 \ .$$

$$r_i(\boldsymbol{w}_\nu, \boldsymbol{x}) = x_i - \hat{a}_1(\boldsymbol{w}_\nu, \boldsymbol{x}) = \frac{x_i s_\mu^2 D_\mu(z) - \overline{\overline{(Cx)}}_\mu}{s_\nu^2 D_\nu(z)}, \qquad \text{where} \quad C_j = \overline{\overline{z^2}} - z_j \overline{\overline{z}} \ .$$

$$r_i(\boldsymbol{w}_\mu, \boldsymbol{x}) = \frac{x_i s_\mu^2 D_\mu(z) - \overline{\overline{(Cx)}}_\mu}{s_\mu^2 D_\mu(z)}$$

- Numerators are identical!
- Numerators are linear combinations of hit coordinates $\Rightarrow$ can replace them to errors and apply the error propagation rules $\Rightarrow$
- Numerators of variances are the same $\Rightarrow$ numerator of root of product is the same as the numerators of residual variances taken separately!

Geometric mean, general expression and expression with with optimal weights

Denote the geometric mean $V_i^{(\mathrm{gm})}(r) = \sqrt{V(r_i(\boldsymbol{w}_\nu, \boldsymbol{x})V(r_i(\boldsymbol{w}_\mu, \boldsymbol{x}))}$.
General expression:

$$V_i^{(\mathrm{gm})} = \frac{s_\mu^4 D_\mu^2(z)V(\epsilon_i) + \sum\limits_{j\neq i} w_{\mu,j}^2 C_j^2 V(\epsilon_j)}{s_\nu^2 D_\nu(z) s_\mu^2 D_\mu(z)} \ .$$

This coincides with inclusive or exclusive residuals with some different factors $\Rightarrow$ can be handled similarly and does not provide additional information.

According to [R.K. Carnegie, et al, NIM A 538(2005)372] and [D.C. Arogancia, et al, NIM A 602(2009)403] the weights w_i should be equal to inverse variance $w_i = 1/V(\epsilon_i)$ (optimal for the least squares method). Then,

$$V_i^{(\mathrm{gm})} = \frac{s_\mu^4 D_\mu^2(z)V(\epsilon_i) + \sum\limits_{j\neq i} w_{\mu,j} C_j^2 w_{\nu,i} V(\epsilon_i)}{s_\nu^2 D_\nu(z) s_\mu^2 D_\mu(z)} =$$

$$\frac{s_\mu^2 D_\mu(z)V(\epsilon_i) + \overline{\overline{z^2}} w_{\nu,i} V(\epsilon_i)}{s_\nu^2 D_\nu(z)} = V(\epsilon_i).$$

This result is beautiful, but useless, because in order to obtain residuals with weights $w_i = 1/V(\epsilon_i)$, we have already to know these very resolutions $\sqrt{V(\epsilon_i)}$, which we want to obtain. Numerical tests show that an iteration procedure with remaking the residuals with previously obtained weights is not useful too.

Geometric mean with unity weights

Another interesting special case is $w_i = 1$, for which

$$V_i^{(\mathrm{gm})}(r) = \frac{\left((N-1)\overline{\overline{z^2}} - \overline{\overline{z}}^2\right)^2 V(\epsilon_i) + \sum\limits_{j\neq i}\left(\overline{\overline{z^2}} - z_j\overline{z}\right)^2 V(\epsilon_j)}{\left(N\overline{\overline{z^2}} - \overline{\overline{z}}^2\right)\left((N-1)\overline{\overline{z^2}} - \overline{\overline{z}}^2\right)} \ .$$

Let us assume that we have a telescope of detectors with the same resolution $V(\epsilon_j) = B$ and we want to study the tested detector i having possibly different resolution. Then, equation simplifies to

$$V_i^{(\mathrm{gm})}(r) = \frac{\left((N-1)\overline{\overline{z^2}} - \overline{\overline{z}}^2\right) V(\epsilon_i) + \overline{\overline{z^2}}B}{\left(N\overline{\overline{z^2}} - \overline{\overline{z}}^2\right)} \ .$$

This allows us to determine $V(\epsilon_i)$ provided that B is known or if B is a function of $V(\epsilon_i)$. If $B = V(\epsilon_i)$

$$V_i^{(\mathrm{gm})}(r) = V(\epsilon_i) \ ,$$

The Monte-Carlo simulations in Ref. [T. Alexopoulos et al., JINST, 9 (2014) P01003] seem to confirm that this formula is correct for equal resolutions and not accurate for non-equal resolutions, but unfortunately this work like many others do not specify which weights were used for track fitting. So we can only assume that the weights were unity by default. My simulations with unity weights bring about to the same conclusion. Refs. [Carnegie,Arogancia] also do not comment on what weights should be used to calculate the residuals in practical applications of the equation $V_i^{(\mathrm{gm})}(r) = V(\epsilon_i)$. Note that the same results can be easily obtained from any single residual:

$$V(\epsilon_i) = \frac{V(r_i)}{\sum_{j=1}^{N} u_{ij}^2}.$$

Reconstruction of resolution, three detecting layers

The H-matrix for three layers is singular for any *given* system of equations for variances, covariances, weights, and z-coordinates of layers.

It is however not easy to prove that this holds for *arbitrary* set of equations and parameters.

"Analytical" proof is unknown, but the singularitry can be proved by CAS:

- All systems of equations are singular.
- Always: $\dfrac{h^{(\nu,\mu)}_{i,j,2}}{h^{(\nu,\mu)}_{i,j,1}} = \dfrac{(z_3-z_1)^2}{(z_3-z_2)^2}$, $\dfrac{h^{(\nu,\mu)}_{i,j,3}}{h^{(\nu,\mu)}_{i,j,1}} = \dfrac{(z_2-z_1)^2}{(z_3-z_2)^2}$.
  1. All weights are cancelled!
  2. All equations are linearly dependent.

  Therefore, only one value can be obtained. For example, for equal gaps: the inaccurate average $\big(V(\epsilon_1) + 4V(\epsilon_2) + V(\epsilon_3)\big)/6$.

But if $z_1 \approx z_2 \approx z_3$, all three resolutions can be easily found.

The same is true if the directions or tracks are known, i.e. measured by a remote detecting payer.

If one can move or rearrange layers, all resolutions can be reconstructed.

The idea to check layer permutations was proposed by N. V. Gruzinsky.

For example, for unity weights;

$V(r_1)$ is obtained after permutation of the first and the second layer,

$V(r_2)$ is obtained without permutations or moving,

$V(r_3)$ is obtained after permutation of the second and the third layer

(the residual in the original second layer is always obtained, but this layer is placed in different places), then

$$H^{-1} = \begin{pmatrix} 10 & -0.5 & -2 \\ -2 & 2.5 & -2 \\ -2 & -0.5 & 10 \end{pmatrix}$$

Reconstruction of resolution, four detecting layers

The H-matrix for four layers is singular for any *given* system of equations for variances, covariances, weights, and z-coordinates of layers.
"Analytical" proof is unknown, but the singularitry can be shown by CAS.
Considerable computational problems for CAS:
too many combinations of equations and parameters.
Need filtering to avoid effectively the same systems of equations, because permutations of variables do not always make the systems of equations different, see backup slide),

After filtering 164 systems of equations remain.

CAS Reduce and GiNaC: All determinants are zero.

This conclusion agrees with result of R. Frühwirth, which is written for a particular case, and disagrees with claims of J. Bortfeldt for another particular case.

Remove any line in any of 164 systems of equatuions.
Compose 3×3 matrix from 3 first columns.
CAS Reduce: All determinants are non-zero.
Therefore all these H-matrices have rank 3,
and that only 3 y-values are linearly independant.

Reconstruction of resolution, four detecting layers, average resolution of layers with fixed positions by matrix algebra

Suppose that there are 3 parameters α_i, that allow one to express the average variance as a linear combination of measured $\boldsymbol{y}$:

$$\forall V(\boldsymbol{\epsilon}) \quad \sum_{i=1}^{3} \alpha_i y_i = \boldsymbol{\alpha}^{\mathrm{pT}} \boldsymbol{y}^{\mathrm{p}} = \boldsymbol{\alpha}^{\mathrm{pT}} H^{\mathrm{p}} V(\boldsymbol{\epsilon}) = \frac{1}{4} \sum_{i=1}^{4} V(\epsilon_i) \quad \Rightarrow \quad H^{\mathrm{pT}} \boldsymbol{\alpha}^{\mathrm{p}} = \frac{1}{4} \mathbf{1} \ .$$

The superscript "p" ("partial") means vectors with 3 components and 3×4 matrices.
Denote by T the 3×3 matrix composed of three first rows of H^{pT}.
The solution of three first equations is

$$\boldsymbol{\alpha}^{\mathrm{p}} = \frac{1}{4} T^{-1} \mathbf{1},$$

CAS "Reduce": none of the solutions of the first three equations satisfies the fourth equation for arbitrary weights and layer positions (z-values).

The exact average resolution cannot be found for arbitrary layer positions.

Symmetry of layer positions: $z_2 - z_1 = z_4 - z_3$
(called "symmetric detector", "symmetric arrangement") $\Rightarrow$
the fourth equation is always satisfied.

The average resolution can be found in symmetric 4-layer detector.

Reconstruction of resolution, four detecting layers, average resolution of layers with fixed positions by matrix algebra (2)

A different type of symmetry: the H-matrix is symmetric with respect to its center:

$$h_{ij} = h_{4-i+1,4-j+1}.$$

$$H^{\mathrm{r}} \begin{pmatrix} V(\epsilon_1) + V(\epsilon_4) \\ V(\epsilon_2) + V(\epsilon_3) \end{pmatrix} = \begin{pmatrix} y_1 + y_4 \\ y_2 + y_3 \end{pmatrix}, \quad \text{where } H^{\mathrm{r}} = \begin{pmatrix} h_{11} + h_{14} & h_{12} + h_{13} \\ h_{21} + h_{24} & h_{22} + h_{23} \end{pmatrix}.$$

If H^{r} is not singular:

$$\begin{pmatrix} (V(\epsilon_1) + V(\epsilon_4))/2 \\ (V(\epsilon_2) + V(\epsilon_3))/2 \end{pmatrix} = \frac{1}{2}(H^{\mathrm{r}})^{-1} \begin{pmatrix} y_1 + y_4 \\ y_2 + y_3 \end{pmatrix},$$

The total average $\frac{1}{4}\sum_{i=1}^{4} V(\epsilon_i)$ is then calculated as the mean of these two averages.

CAS "Reduce": the H^{r}-matrix is not singular for some general set of symmetric H-matrices written for symmetric detectors.

The most obvious and useful choice of such a matrix for $\boldsymbol{y}$ consisting of variances measured sequentially in layers 1, 2, 3, and 4:

$$\boldsymbol{y} = \begin{pmatrix} V(r_{1,1}) \\ V(r_{2,2}) \\ V(r_{3,3}) \\ V(r_{4,4}) \end{pmatrix},$$

and for symmetric weights: $w_{i,k} = w_{4-i+1,4-k+1}$. Then the H-matrix is symmetric with respect to its center.

Simplification of H-matrix by row reduction

Any H-matrix can be "regularized" unsing a simple row reduction method.
Recall the row reduction method:
The row reduction: subtraction of a line m from all other lines with a factor h_{nm}/h_{mm}.
Perform this procedure for $m = 2$, $m = 3$, and $m = 1$.
In addition, subtract the second line with factor presented by the ratio of *new values* in the last column h_{34}/h_{24} from the third line.
Each step $\equiv$ multiplication of the H-matrix by another matrix: $GHV(\epsilon) = G\boldsymbol{y}$,
An unexpected property of this matrix:

all k-values are identical and independent on the weights.

$$GH = \begin{pmatrix} c_{11} & 0 & 0 & c_{11}k_1 \\ 0 & c_{22} & 0 & c_{22}k_2 \\ 0 & c_{32} & c_{32}k_3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$k_1 = \frac{(z_2 - z_1)(z_3 - z_1)}{(z_4 - z_2)(z_4 - z_3)},$$

$$k_2 = -\frac{(z_2 - z_1)(z_3 - z_2)}{(z_4 - z_3)(z_4 - z_1)},$$

$$k_3 = \frac{(z_2 - z_1)(z_4 - z_2)}{(z_3 - z_1)(z_4 - z_3)}.$$

This is obtained for particular equations and weights and proven algebraically provided that the rank of H-matrix is not greater than 3. Otherise the maximal rank should be 4:
Suppose that there are different G_1H_1 and G_2H_2 and least one of k_i is different in them.

$$\begin{pmatrix} G_1H_1 \\ G_2H_2 \end{pmatrix} = \begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix} \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}.$$

The rank of the left-hand side is 4 $\Rightarrow$ the rank of $\begin{pmatrix} H_1 \\ H_2 \end{pmatrix}$ should not be less than 4 which contraducts to calculations with Reduce and GiNaC.

Therefore, the values of k_i cannot differ in any G_1H_1 and G_2H_2.

Properties of reduced H-matrix

Repetition:

$$GH = \begin{pmatrix} c_{11} & 0 & 0 & c_{11}k_1 \\ 0 & c_{22} & 0 & c_{22}k_2 \\ 0 & c_{32} & c_{32}k_3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$k_1 = \frac{(z_2 - z_1)(z_3 - z_1)}{(z_4 - z_2)(z_4 - z_3)},$$

$$k_2 = -\frac{(z_2 - z_1)(z_3 - z_2)}{(z_4 - z_3)(z_4 - z_1)},$$

$$k_3 = \frac{(z_2 - z_1)(z_4 - z_2)}{(z_3 - z_1)(z_4 - z_3)}.$$

The properties:

- $\forall i < j \; z_i < z_j$ (if z-values are sorted in ascending order), then $k_1 > 0$, $k_3 > 0$ and $k_2 < 0$.
- For the same gaps between layers and $\forall i < j \; z_i < z_j$: $k_1 = k_3 = 1, \; k_2 = -1/3$.
- If gaps are not the same, $\forall i < j \; z_i < z_j$, and if the detector is symmetric: $z_2 - z_1 = z_4 - z_3$, than $k_1 = k_3 = 1, \; -1 < k_2 < 0$.
- Always: $h_{i4} = k_1 h_{i1} + k_2 h_{i2} - k_2/k_3 h_{i3}$.

Average resolutions of layers with symmetric arrangement

Assuming symmetric arrangement $z_2 - z_1 = z_4 - z_3$ with z-values sorted in ascending order $\forall i < j \; z_i < z_j$.
For $k_1 = k_3 = 1, \; -1 < k_2 < 0$:

$$GH = \begin{pmatrix} c_{11} & 0 & 0 & c_{11} \\ 0 & c_{22} & 0 & c_{22}k_2 \\ 0 & c_{32} & c_{32} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, GHV(\boldsymbol{\epsilon}) = G\boldsymbol{y} \Rightarrow$$

$$\frac{V(\epsilon_1) + V(\epsilon_4)}{2} = \frac{[G\boldsymbol{y}]_1}{2c_{11}}, \quad \frac{V(\epsilon_2) + V(\epsilon_3)}{2} = \frac{[G\boldsymbol{y}]_3}{2c_{32}}$$

$$\frac{1}{4}\sum_{i=1}^{4} V(\epsilon_i) = \frac{1}{4}\left(\frac{[G\boldsymbol{y}]_1}{c_{11}} + \frac{[G\boldsymbol{y}]_3}{c_{32}}\right)$$

Here $[G\boldsymbol{y}]_1$ and $[G\boldsymbol{y}]_3$ are the first and the third component of the vector $G\boldsymbol{y}$.

The resolutions of layers provided that the resolution of two given layers are equal

For symmetric detectors, add the first row of the GH-matrix with a factor $-k_2 c_{22}/(2c_{11})$ to the second row.

$$GH = \begin{pmatrix} c_{11} & 0 & 0 & c_{11} \\ -k_2 c_{22}/2 & c_{22} & 0 & c_{22} k_2/2 \\ 0 & c_{32} & c_{32} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

For equal gaps $k_2/2 = -1/6$.
If $V(\epsilon_1) \approx V(\epsilon_4) \Rightarrow$
$[GHV(\boldsymbol{\epsilon})]_2 \approx c_{22} V(\epsilon_2) \Rightarrow$
$V(\epsilon_2) \approx \frac{[G\boldsymbol{y}]_2}{c_{22}} + \frac{[G\boldsymbol{y}]_1}{6c_{11}}$,
$V(\epsilon_3) \approx \frac{[G\boldsymbol{y}]_3}{c_{32}} - V(\epsilon_2) =$
$= -\frac{[G\boldsymbol{y}]_1}{6c_{11}} - \frac{[G\boldsymbol{y}]_2}{c_{22}} + \frac{[G\boldsymbol{y}]_3}{c_{32}}$

General solution for equal layers 1 and 4 (for example; the superscript (r) means reconstructed):

$$\widehat{V}(\epsilon_{14}) = \frac{[G\boldsymbol{y}]_1}{c_{11}(1+k_1)} ,$$

$$\widehat{V}(\epsilon_2) = -k_2 \frac{[G\boldsymbol{y}]_1}{c_{11}(1+k_1)} + \frac{[G\boldsymbol{y}]_2}{c_{22}} ,$$

$$\widehat{V}(\epsilon_3) = \frac{1}{k_3}\left(k_2 \frac{[G\boldsymbol{y}]_1}{c_{11}(1+k_1)} - \frac{[G\boldsymbol{y}]_2}{c_{22}} + \frac{[G\boldsymbol{y}]_3}{c_{32}}\right)$$

The resolutions of layers provided that the resolution of two given layers are equal (2)

Assume that $\sigma(\epsilon_i) \approx \sigma(\epsilon_j)$, but $\sigma(\epsilon_i) - \sigma(\epsilon_j) = 2\tau$, where τ is some small number; the second form is for equal gaps:

For identical layers $i = 1$ and $j = 4$:

$$\frac{\mathrm{d}\widehat{\sigma}(\epsilon_{14})}{\mathrm{d}\tau} = \frac{k_1 - 1}{k_1 + 1}\,\frac{\sigma(\epsilon_{14})}{\widehat{\sigma}(\epsilon_{14})} = 0,$$

$$\frac{\mathrm{d}\widehat{\sigma}(\epsilon_2)}{\mathrm{d}\tau} = \frac{2k_2}{k_1 + 1}\,\frac{\sigma(\epsilon_{14})}{\widehat{\sigma}(\epsilon_2)} = -\frac{1}{3}\,\frac{\sigma(\epsilon_{14})}{\widehat{\sigma}(\epsilon_2)}\ ,$$

$$\frac{\mathrm{d}\widehat{\sigma}(\epsilon_3)}{\mathrm{d}\tau} = -\frac{2k_2}{k_3(k_1 + 1)}\,\frac{\sigma(\epsilon_{14})}{\widehat{\sigma}(\epsilon_3)} = \frac{1}{3}\,\frac{\sigma(\epsilon_{14})}{\widehat{\sigma}(\epsilon_3)}\ .$$

For identical layers $i = 2$ and $j = 3$:

$$\frac{\mathrm{d}\widehat{\sigma}(\epsilon_{23})}{\mathrm{d}\tau} = \frac{k_3 - 1}{k_3 + 1}\,\frac{\sigma(\epsilon_{23})}{\widehat{\sigma}(\epsilon_{23})} = 0,$$

$$\frac{\mathrm{d}\widehat{\sigma}(\epsilon_1)}{\mathrm{d}\tau} = \frac{2k_1k_3}{k_2(k_3 + 1)}\,\frac{\sigma(\epsilon_{23})}{\widehat{\sigma}(\epsilon_1)} = -3\frac{\sigma(\epsilon_{23})}{\widehat{\sigma}(\epsilon_1)}\ ,$$

$$\frac{\mathrm{d}\widehat{\sigma}(\epsilon_4)}{\mathrm{d}\tau} = -\frac{2k_3}{k_2(k_3 + 1)}\,\frac{\sigma(\epsilon_{23})}{\widehat{\sigma}(\epsilon_4)} = 3\frac{\sigma(\epsilon_{23})}{\widehat{\sigma}(\epsilon_4)}\ .$$

For supposed equal layers i and j:

	$i=1$ $j=4$	$i=2$ $j=3$	$i=1$ $j=2$	$i=1$ $j=3$	$i=2$ $j=4$	$i=3$ $j=4$
$\sigma'_\tau(\epsilon_1)$	0	-3	1/2	2	3	3/2
$\sigma'_\tau(\epsilon_2)$	−1/3	0	1/2	-1	-2	−1/2
$\sigma'_\tau(\epsilon_3)$	1/3	0	1/2	2	1	−1/2
$\sigma'_\tau(\epsilon_4)$	0	3	−3/2	-3	-2	−1/2

Allocate supposingly equal layers outside —
and obtain almost exact resolutions of internal layers!

Average resolutions of layers with asymmetric arrangement

If the detector is not perfectly symmetric, but the offset is small due to misalignment, etc., no exact average value, but how about the approximate estimate?
Instead of

$$\frac{V(\epsilon_1)+V(\epsilon_4)}{2}=\frac{[G\boldsymbol{y}]_1}{2c_{11}},\quad \frac{V(\epsilon_2)+V(\epsilon_3)}{2}=\frac{[G\boldsymbol{y}]_3}{2c_{32}}$$

$$\frac{1}{4}\sum_{i=1}^{4}V(\epsilon_i)=\frac{1}{4}\Big(\frac{[G\boldsymbol{y}]_1}{c_{11}}+\frac{[G\boldsymbol{y}]_3}{c_{32}}\Big)$$

a weighted averages $\overline{v}=\sum_{i=1}^{N}w_i v_i/s$, $s=\sum_{i=1}^{N}w_i=1$.
$\gamma_1 c_{11}V(\epsilon_1)+\gamma_1 c_{11}k_1 V(\epsilon_4)=\gamma_1[G\boldsymbol{y}]_1$
$\gamma_1 : \gamma_1 c_{11}+\gamma_1 c_{11}k_1=\gamma_1 c_{11}(1+k_1)=1\Rightarrow\gamma_1=1/(c_{11}(1+k_1))$ and
(with similar derivation for the third row):

$$\widehat{\frac{V(\epsilon_1)+V(\epsilon_4)}{2}}=\frac{[G\boldsymbol{y}]_1}{c_{11}(1+k_1)},\quad \widehat{\frac{V(\epsilon_2)+V(\epsilon_3)}{2}}=\frac{[G\boldsymbol{y}]_3}{c_{32}(1+k_3)}$$

$$\widehat{\frac{1}{4}\sum_{i=1}^{4}V(\epsilon_i)}=\frac{[G\boldsymbol{y}[_1}{2c_{11}(1+k_1)}+\frac{[G\boldsymbol{y}]_3}{2c_{32}(1+k_3)}.$$

For asymmetric detectors slightly more accurate estimates by fits of vector $\boldsymbol{\alpha}$ in equations like $\sum_{i=1}^{4}\alpha_i V(r_i)=\boldsymbol{\alpha}^{\intercal}HV(\boldsymbol{\epsilon})=\frac{1}{4}\sum_{i=1}^{4}V(\epsilon_i)$, $H^{\intercal}\boldsymbol{\alpha}=\frac{1}{4}\mathbf{1}$, denote $\boldsymbol{\beta}=H^{\mathrm{T}}\boldsymbol{\alpha}$.

For example, for the total average to minimize: $S=\sum_{i=1}^{4}(\beta_i-1/4)^2$, provided that $\sum_{i=1}^{4}\beta_i=1$.

Moving layers

Assume that the gaps are equal.
For example, for unity weights;
$V(r_2)$ and $V(r_3)$ are obtained with initial allocation of layers,
$V(r_1)$ and $V(r_4)$ be obtained after the permutation of the first and the last layer.
(The residual $V(r_1)$ is actually measured in the fourth layer allocated in the position of the first layer, and vice versa.)

$$H^{-1} = \begin{pmatrix} 13/8 & -11/24 & -91/24 & 93/8 \\ -11/24 & 157/72 & 77/72 & -91/24 \\ -91/24 & 77/72 & 157/72 & -11/24 \\ 93/8 & -91/24 & -11/24 & 13/8 \end{pmatrix}$$

Reconstruction of resolution, five detecting layers

- For any 4 detector layers we can write 3 independent equations.
- Consider two combinations of 4 layers taken from 5 layers, such that the second combination includes the layer which was not included in the first one.
- For the second combinations of layers write equations that include the replaced layer.
- Then any equation of the second combination is not linearly dependent on equations of the first combination because of the appearance of extra layer.
- Therefore we can obtain 5 and even more independent equations.

For example, two-layer straight lines provide 195 non-singular systems of equations.
Obviously, there are much more independent equations for 6- and more-layer detectors.
It is reasonable to use simple "exclusive" residuals with number of equations being equal to the number of layers.
The example of inverse matrix for equally spaced five layers:

$$H^{-1} = \begin{pmatrix} 1.746 & -1.749 & -0.187 & 0.701 & -0.653 \\ -0.571 & 1.578 & -0.022 & -0.260 & 0.229 \\ -0.046 & -0.017 & 1.026 & -0.017 & -0.047 \\ 0.228 & -0.260 & -0.022 & 1.578 & -0.571 \\ -0.654 & 0.701 & -0.188 & -1.748 & 1.746 \end{pmatrix}.$$

Conclusion

1. A general method for reconstructing individual resolutions of detecting layers in multi-layer detectors is developed.
2. The individual layer resolutions can be obtained for 3- and 4-layer detectors only if there is a possibility to move the layers.
3. If the layers cannot be moved, but the 4-layer detector is symmetric, the following values can be obtained:
   a) The average squared resolution of four layers;
   b) The average squared resolutions of layers 1 and 4, as well as 2 and 3;
   c) If the resolutions of layers 1 and 4 are assumed to be equal, the individual resolutions of all layers (assuming the first and the fourth identical) can be obtained.
4. If the symmetry is slightly violated in the 4-layer detector, approximate estimates of all values mentioned in the previous item can be obtained.
5. The individual resolutions can be obtained for 5- and more-layer detectors.
6. All these results can be calculated by either residuals or geometrical means of inclusive and exclusive residuals. The geometric means of residuals, as well as correlations of residuals, do not produce any additional information. The geometric means obtained with unity weights are equal to the layer resolution only if the detector layers have the same resolution.

Backup slides: Priciples of proofs by CAS, used in this research

- If a statement $A(x_1, x_2)$ (i.e. equation) is true $\forall$ x_1 and x_2 from any set, for example $\mathbb{R}$, then $A(x_2, x_1)$ is also true.
- If a statement $A(x_1, x_1.x_2)$ is true $\forall$ x_1 and x_2 from any set, $A(x_2, x_2.x_1)$ is also true, but $A(x_1, x_2.x_2)$ may not be true.

The result is not changed if different variables are replaced by other different variables; identical variables are replaced by other identical variables.

There is no need to test all identical combinations.

Assign the "permanent" unique weight index for each residual; for three-layer cases:

$$H = \begin{pmatrix} h^{(1,2)}_{i_1,j_1,1} & h^{(1,2)}_{i_1,j_1,2} & h^{(1,2)}_{i_1,j_1,3} \\ h^{(3,4)}_{i_2,j_2,1} & h^{(3,4)}_{i_2,j_2,2} & h^{(3,4)}_{i_2,j_2,3} \\ h^{(5,6)}_{i_3,j_3,1} & h^{(5,6)}_{i_3,j_3,2} & h^{(5,6)}_{i_3,j_3,3} \end{pmatrix} = \begin{pmatrix} u_{\nu_1,i_1 1}u_{\mu_1,j_1 1} & u_{\nu_1,i_1 2}u_{\mu_1,j_1 2} & u_{\nu_1,i_1 3}u_{\mu_1,j_1 3} \\ u_{\nu_2,i_2 1}u_{\mu_2,j_2 1} & u_{\nu_2,i_2 2}u_{\mu_2,j_2 2} & u_{\nu_2,i_2 3}u_{\mu_2,j_2 3} \\ u_{\nu_3,i_3 1}u_{\mu_3,j_3 1} & u_{\nu_3,i_3 2}u_{\mu_3,j_3 2} & u_{\nu_3,i_3 3}u_{\mu_3,j_3 3} \end{pmatrix}$$

In the cases, where z-values are not restricted, the indexes may be incremented without gaps; the selection criteria (for integer $n \in \mathbb{Z}$):

$i_1 = 1 \wedge \left(\forall n \in [1, N]\ (j_n = i_n \vee j_n = i_n + 1)\right) \wedge \left(\forall n \in [2, N]\ i_n \in [i_{n-1}, \max\{i_{n-1}, j_{n-1}\} + 1]\right)$

For 4-layer detectors, from $16^4 = 65536$ systems of equations, it leaves only 164 systems!!!

For symmetric detectors: $\left(\forall n \in [1, N]\ i_n \leqslant j_n\right) \wedge \left(\forall n \in [2, N]\ (i_n > i_{n-1} \vee (i_n = i_{n-1} \wedge j_n \geqslant j_{n-1}))\right) \wedge 1 \leqslant \imath_1 \leqslant N - \max\limits_{n \in [1,N]} \{j_n\} + 1$ leaves 399 systems.

This 4×4 H-matrix is symmetric with respect to its center if the detector is symmetric; if the weights are symmetric: $\forall \nu \in [5, 8] \wedge \forall i \in [1, 4]\ \boldsymbol{w}_{\nu,i} = \boldsymbol{w}_{9-\nu,5-i}$; and if i_n and j_n (assuming that $i_n \leqslant j_n$) are symmetric: $(i_1 = 5 - j_4 \wedge j_1 = 5 - i_4) \wedge (i_2 = 5 - j_3 \wedge j_2 = 5 - i_3)$. There are 18 systems that satisfy these conditions.