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Improved renormalization of lattice operators: A critical reappraisal

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Abstract. We systematically examine various proposals which aim at increasing the accuracy in the determination of the renormalization of two-fermion lattice operators. We concentrate on three finite quantities which are particularly suitable for our study: the renormalization constants of the vector and axial currents and the ratio of the renormalization constants of the scalar and pseudoscalar densities. We calculate these quantities in boosted perturbation theory, with several running boosted couplings, at the "optimal" scale q^*. We find that the results of boosted perturbation theory are usually (but not always) in better agreement with non-perturbative determinations of the renormalization constants than those obtained with standard perturbation theory. The finite renormalization constants of two-fermion lattice operators are also obtained non-perturbatively, using Ward Identities, both with the Wilson and the tree-level Clover improved actions, at fixed cutoff ($\beta = 6.4$ and 6.0 respectively). In order to amplify finite cutoff effects, the quark masses (in lattice units) are varied in a large interval $0 \lesssim am \lesssim 1$. We find that discretization effects are always large with the Wilson action, despite our relatively small value of the lattice spacing ($a^{-1} \simeq 3.7$ GeV). With the Clover action discretization errors are significantly reduced at small quark mass, even though our lattice spacing is larger ($a^{-1} \simeq 2$ GeV). However, these errors remain substantial in the heavy quark region. We have implemented a proposal for reducing $\mathcal{O}(am)$ effects, which consists in matching the lattice quantities to their continuum counterparts in the free theory. We find that this approach still leaves appreciable, mass dependent, discretization effects.

1 Introduction

One of the major goals of lattice QCD computations is the non-perturbative evaluation of hadronic Matrix Elements (ME's) which are relevant for strong and electroweak interactions. On the lattice, the ME's of bare operators are computed at finite UV cutoff a^{-1} (a is the lattice spacing) by simulating QCD at fixed bare coupling $\beta = 6/g_0^2$. These ME's are functions of the finite cutoff. In order to extract physical information from them, the lattice operators require renormalization. For example, the ρ-meson decay constant f_ρ^{-1} is obtained from the ME $\langle 0|V_k(0)|\rho_r\rangle$ as follows:

$$\epsilon_k^r \frac{m_\rho^2}{f_\rho} = Z_V \langle 0|V_k(0)|\rho_r\rangle \tag{1}$$

where m_ρ is the meson mass and ϵ_k^r is its polarization vector. V_k is a spatial component of the bare (lattice) vector current and Z_V is its renormalization constant (RC). Z_V connects the lattice theory at finite cutoff to the renormalized continuum theory. Analogously, the π-meson's decay constant f_π is determined from the ME of the axial current through:

$$if_\pi m_\pi = Z_A \langle 0|A_0(0)|\pi\rangle \tag{2}$$

where m_π is the pion mass, A_0 is the time component of the bare (lattice) axial current and Z_A is its RC. These are specific examples in which operator renormalization is an essential step for passing from lattice to continuum physics. This renormalization is an inevitable requirement in those cases where the ME's of bare operators diverge in the continuum limit (e.g. the ME's of the $\Delta I = 1/2$ four-fermion operators).

So far, three methods have been implemented to the computation of the Z's:

- Lattice Perturbation Theory (PT) [1]–[10]
- Non-perturbative use of Ward Identities (WI's), either on hadronic states [11]–[20] or on quark states [17,19, 21].
- Non-perturbative (NP) renormalization on external quark and/or gluon states [22]–[27].

Lattice PT can be applied to the calculation of finite and logarithmically divergent RC's[1]. The WI method is only applicable to finite RC's, whereas the NP one can be applied to all cases. The RC's, calculated to $\mathcal{O}(g_0^{2n})$ in PT, suffer from $\mathcal{O}(g_0^{2(n+1)})$ systematic errors (in most cases $n = 1$). On the other hand, non-perturbative determinations of the RC's (WI or NP method), are subject to $\mathcal{O}(a)$ systematic errors, since they are obtained from correlation functions computed at finite cutoff. Thus, physical quantities like those of (1) and (2), will suffer from $\mathcal{O}(g_0^{2(n+1)})$ and $\mathcal{O}(a)$ systematic errors if the RC's are obtained in PT, and from $\mathcal{O}(a)$ errors if they are computed non-perturbatively. Therefore WI or NP estimates of the RC's are clearly preferable in principle.

In recent years, there have been considerable efforts in refining the perturbative computation of the RC's. Boosted PT (BPT) has been extensively explored by Lepage and Mackenzie [29]. It consists in judicious choices of the expansion parameter (motivated by the behaviour of the resulting perturbative series, by tadpole resummation arguments and even by Mean Field Theory) which should bring the perturbative estimate of the RC's closer to their actual values. Even though the error remains $\mathcal{O}(g_0^{2(n+1)})$, any of the several choices of BPT, proposed in [29], might improve the asymptotic convergence of the perturbative series.

On the non-perturbative side, Symanzik-type improved actions and operators have been constructed, in order to systematically reduce the finite cutoff error inherent in numerical computations. The implementation of the tree-level improved Clover action and operators [30, 31] reduces the discretization errors from $\mathcal{O}(a)$ to $\mathcal{O}(g_0^2 a)$ (more precisely, it removes all $\mathcal{O}(a g_0^{2n} \ln^n a)$ systematic errors, which are effectively $\mathcal{O}(a)$ in the scaling limit, $g_0^2 \sim 1/\ln a$). Recently, a non-perturbative extension of the tree-level Clover improvement programme has been proposed in [32], which corrects up to $\mathcal{O}(a^2)$ errors. The general proposal of these works is directly applicable in the chiral limit (with the exception of the improvement of the vector current which has also been implemented at non-zero quark mass; see [33]). Another approach to the non-perturbative improvement programme, which can also be applied with finite values of the quark masses, can be found in in [34,35].

Regarding the RC's, the main achievement of the tree-level improvement programme has been the reduction of discretization errors from about 20% to about 5% in computations with light quark masses [15]. The situation is more problematic with heavy quark masses, say $am \gtrsim 0.4$. Even with a tree-level improved action, the dominant $\mathcal{O}(g_0^2 am)$ error is appreciable. The most direct way of reducing these errors consists in reducing the lattice spacing (by going to larger β values), but the increase in computational cost becomes prohibitive. Alternatively, one can try to reduce systematically these errors at fixed cutoff. The proposal of [36]–[39] is such an effort. It consists in matching lattice quantities to their continuum counterparts at vanishing spatial momentum, ap, and large finite masses, am. The aim is to eventually remove all am corrections, leaving predominantly $\mathcal{O}(ap)$ errors. This program has been explicitly carried out only in the free theory; we refer to it as the KLM approach. Speculations about higher order extensions can be found in [39].

This work is a thorough comparison of the various estimates (perturbative and non-perturbative) of the RC's of two-fermion lattice operators. We have tried to be exhaustive in investigating the numerous improvement proposals found in the literature. In particular:

- We have concentrated on the finite RC's of the vector and axial currents (Z_V and Z_A) and the finite ratio Z_S/Z_P of the RC's of the scalar and pseudoscalar densities.
- We have examined the influence of boosting to the 1-loop perturbative estimates of the above quantities. Several boosting recipes of [29] have been applied. The "optimal" scales of the boosted running coupling (the q^* of [29]) has been obtained for Z_V, Z_A and Z_S/Z_P. The q^* values can be found in Table 1. We find that all recipes of BPT give the same results within a few per cent (10% in the worst case). These results are summarized in Table 2. Moreover, all BPT results differ from those obtained in SPT by up to 20%. When compared with the RC's obtained from WI's (see Table 3), BPT results are usually in better agreement than the SPT ones.
- The RC's Z_V, Z_A and the ratio Z_S/Z_P have subsequently been computed non-perturbatively, using the WI method, for several quark masses. Both the Wilson and the tree-level Clover actions have been implemented. Discretization errors at small quark mass, being $\mathcal{O}(a)$ tend to be about 20% - 30% in the Wilson case. In the tree-level Clover case, being $\mathcal{O}(g_0^2 a)$, these errors drop below 5%. However, for masses $ma \gtrsim 0.4$, which are used in current lattice QCD simulations, they grow beyond 10%. By extrapolating our data in the quark mass, we obtain extremely consistent Clover results for Z_V and Z_A at zero quark mass, shown in Table 3. We have also checked that it is not possible to optimize these results by implementing different choices of improved operators, based on a different arbitrary admixture of the so-called $\not{\partial}$ and m_0 improving field rotations (see Subsect. 5.2 for the definitions of these field rotations).
- In order to control discretization errors of increasing quark mass, we have applied the tree-level KLM matching prescription of [36]–[39]. In the Wilson case, the large systematic error due to $\mathcal{O}(am)$ effects is significantly reduced in some cases (see Fig. 1). There are cases, however, for which the KLM prescription fails to correct large $\mathcal{O}(am)$ effects (see Fig. 2). Moreover, a similar analysis, applied to the Clover case, revealed that KLM corrections fail to improve the situation. This is because the Clover correlation functions we have examined are characterized by large $\mathcal{O}(g_0^2 am)$

[1] PT cannot be applied to power divergent RC's, because of the possible presence of finite non-perturbative contributions (see, for example, [25,28] and references therein)

errors in the region of heavy quark masses. Thus, the KLM corrections, being of $\mathcal{O}(a^2)$, cannot absorb these discretization errors.

The present work is intended as a thorough comparison of the various improvement schemes which have been widely used in present-day, state-of-the-art lattice QCD phenomenology studies. As such, it has not dealt with the more recent non-perturbative Clover improvement of [32, 34,35], which appears to be the most promising way to reduce finite cutoff effects.

The paper is organized as follows. In Sect. 2 we give some basic definitions. In Sect. 3 we review the perturbative calculation of the RC's of two-fermion lattice operators. In Subsect. 3.1 we discuss BPT and compare results for the RC's obtained from various boosting recipes. We also present the "optimal scales" q^* of [29] for the operators of interest (some technical subtleties of this calculation can be found in the Appendix). In Sect. 4 we review the well known vector and axial WI's on the lattice, with particular emphasis on the conditions under which they can be used to obtain the RC's. We have gathered in this section all WI's which are subsequently used in Sects. 5 and 7, where we discuss the practical implementation of the WI determination of RC's and present our results for Z_V, Z_A and Z_S/Z_P. In Sect. 6 we review the tree-level factors of the KLM-procedure for the Wilson correlation functions and derive their Clover counterparts. Finally, in Sect. 8 we present our conclusions.

2 Definitions and generalities on RC's

In this section we define the quantities of interest to this paper. Besides fixing our notation, we also review some general properties of the RC's of lattice operators, with particular attention to their dependence on the theory's parameters and renormalization scale.

We consider bilinear quark operators of the form:

$$O^a_\Gamma(x) = \bar\psi(x)\Gamma\frac{\lambda^a}{2}\psi(x) \tag{3}$$

where $\psi(x)$ is the quark field, Γ denotes a generic Dirac matrix and $\lambda^a/2$ are the generators of the $SU(N_f)$ flavour group in the fundamental representation ($a = 1, \ldots, N_f^2 - 1$). These generators satisfy:

$$\begin{aligned} &Tr[\lambda^a\lambda^b] = 2\delta^{ab} \\ &\left[\frac{\lambda^f}{2}, \frac{\lambda^g}{2}\right] = if^{fgh}\frac{\lambda^h}{2} \\ &\left\{\frac{\lambda^f}{2}, \frac{\lambda^g}{2}\right\} = d^{fgh}\frac{\lambda^h}{2} + \frac{\delta^{fg}}{N_f}\,I \end{aligned} \tag{4}$$

where I represents the identity. Specific bilinear operators will be denoted according to their Lorentz group transformations: the scalar and pseudoscalar densities are denoted by $S^a(x) = \bar\psi(x)\frac{\lambda^a}{2}\psi(x)$ and $P^a(x) = \bar\psi(x)\frac{\lambda^a}{2}\gamma_5\psi(x)$ respectively, whereas the local vector and axial vector currents by $V^a_\mu(x) = \bar\psi(x)\frac{\lambda^a}{2}\gamma_\mu\psi(x)$ and $A^a_\mu(x) = \bar\psi(x)\frac{\lambda^a}{2}\gamma_\mu\gamma_5\psi(x)$. Only such non-singlet bilinear quark operators will be considered in this paper. In the case of the vector and axial currents, we will also be using their point-split discretizations.

We will deal with operator insertions in n-point Green's functions, with the operator usually inserted at the origin. For example the insertion of O_Γ in the 2-point Green's function is

$$G_O(x_1, x_2) = \langle\psi(x_1)O^a_\Gamma(0)\bar\psi(x_2)\rangle \tag{5}$$

In momentum space the insertion becomes

$$\begin{aligned} G_O(p_1, p_2) = \int dx_1 dx_2 \exp(ip_1x_1) \\ \times \exp(-ip_2x_2)G_O(x_1, x_2) \end{aligned} \tag{6}$$

and the corresponding amputated Green's function is given by

$$\Lambda_O(p_1, p_2) = \mathcal{S}^{-1}(p_1)G_O(p_1, p_2)\mathcal{S}^{-1}(p_2) \tag{7}$$

where $\mathcal{S}(x_1 - x_2) = \langle\psi(x_1)\bar\psi(x_2)\rangle$ is the quark propagator and $\mathcal{S}(p)$ its momentum space counterpart. We will also be using Green's functions between external hadronic states. These are defined in an analogous fashion. For example, the 3-point correlation function $\langle P(x_1)V_\mu(0)P^\dagger(x_2)\rangle$ is a correlation function of three bilinear operators. From its Fourier transform in 3-dimensional space, and with the aid of the spectral representation formula at large Euclidean time separations, the on-shell hadronic ME $\langle\pi(\mathbf{p_1})|V_\mu|\pi(\mathbf{p_2})\rangle$ can be readily extracted.

Everything defined so far is a bare quantity, which we assume to be defined in the lattice regularization[2]. We opt for the lattice regularization scheme proposed by Wilson which consists of a gluonic action [40],

$$S_g = -\frac{6}{g_0^2}\sum_P Tr\left[U_P + U_P^\dagger\right] \tag{8}$$

and a fermionic part [41],

$$\begin{aligned} S_f = -a^4\sum_{x,\mu}\frac{1}{2a}\big[&\bar\psi(x)(1-\gamma_\mu)U_\mu(x)\psi(x+\mu) \\ &+\bar\psi(x+\mu)(1+\gamma_\mu)U^\dagger_\mu(x)\psi(x)\big] \\ &+a^4\sum_x\bar\psi(x)(M_0 + \frac{4}{a})\psi(x) \end{aligned} \tag{9}$$

where, in standard notation, U_P is the Wilson plaquette, $U_\mu(x)$ is the lattice gauge link and g_0 the bare coupling constant. The fermion field is a vector in flavour space. Its components will be denoted by u, d, s etc, or by ψ_1, ψ_2, ψ_3, etc. according to notational convenience. The diagonal

[2] This means that the integrals of (6) are really sums ($a^8\sum_{x_1,x_2}$) which run over all lattice sites, labelled by x_1, x_2, etc. The use of integrals instead of sums and Dirac functions instead of Kronecker symbols is a question of notational convenience, which should be transparent

bare mass matrix is denoted by M_0 and its elements by m_{01}, m_{02}, m_{03} etc. (or m_{0u}, m_{0d}, m_{0s} etc.). We use m_0 in the mass degenerate case. As is well known, for Wilson fermions, the quark mass, besides a multiplicative renormalization, is also subject to an additive one. Defining $m = m_0 - m_C$, the renormalized mass will be given by:

$$m_R = Z_m m = Z_m[m_0 - m_C] \tag{10}$$

with Z_m the multiplicative RC. The chiral limit is then $m \to 0$; i.e. $m_0 \to m_C$. At tree-level, $m_C = -4/a$. For the diagonal elements of the matrix $M_0 - m_C$ we will use either m_1, m_2, m_3 etc., or m_u, m_d, m_s etc., according to notational convenience.

All operators studied in this work are subject to multiplicative renormalization; cases involving mixing of equal- or lower-dimensional operators (e.g. the four-fermion operators driving the $\Delta S = 2$ or $\Delta I = 1/2$ transitions) will not be considered. Thus, the renormalized operator is given by

$$\hat{O}_\Gamma(g_R^2, m_R, \mu) = \lim_{a\to 0}[Z_O(g_0^2, a, m, \mu) O_\Gamma(g_0^2, m, a)] \tag{11}$$

where g_R is the renormalized gauge coupling and μ the renormalization scale. Note that in the bare quantities, the bare mass m_0 has been traded off for the more convenient (from the point of view of the chiral limit) subtracted mass m. A specific, if arbitrary, choice of renormalization scheme and scale are implied. Suppressing the dependence on couplings, masses, scales and cutoffs for notational simplicity, we list other relationships of interest between bare and renormalized quantities (denoted in general by "hats"): the renormalized quark propagator is

$$\hat{\mathcal{S}}(p) = \lim_{a\to 0}[Z_\psi \mathcal{S}(p)] \tag{12}$$

where $Z_\psi^{1/2}$ is the quark field RC. From the above definitions of Z_O and Z_ψ, with the aid of (5)–(7), we find for the renormalized Green's functions:

$$\begin{aligned} \hat{G}_O(p_1, p_2) &= \lim_{a\to 0}[Z_\psi Z_O G_O(p_1, p_2)] \\ \hat{\Lambda}_O(p_1, p_2) &= \lim_{a\to 0}[Z_\psi^{-1} Z_O \Lambda_O(p_1, p_2)] \end{aligned} \tag{13}$$

Let us now examine the dependence of the RC's Z_O on the bare parameters of the action. Multiplicative Z_O's are dimensionless quantities. Consequently, their functional dependence on the bare parameters and the cutoff can in principle be only of the form $Z_O(g_0^2, am, a\mu)$. For the specific case of finite RC's (e.g. those of the vector and axial currents) there is no dependence on the renormalization scale μ. In other words, the anomalous dimension is zero; this will be discussed in detail in the following section. Therefore, we are left with a possible dependence of the form $Z_O(g_0^2, am)$. However, there can be no dependence of Z_O on am. Regular terms such as positive powers of am, which may appear in the calculation of the bare Green's function Λ_O, will drop out of Z_O, since RC's are strictly defined in the continuum limit $a \to 0$; c.f. (13). This leaves us with singular terms, such as $\ln(am)$ and negative powers of am. But since we require the existence of the chiral limit ($m \to 0$) in QCD, such a dependence must also be ruled out. This exhausts all possibilities, and renders Z_O independent of am. In conclusion, the finite RC's Z_V, Z_A and the ratio Z_S/Z_P, examined in this paper, are of the form $Z_O(g_0^2)$.[3]

3 Evaluation of RC's from perturbation theory

In this section we review the basic concepts of the perturbative evaluation of the RC's of lattice operators and examine the proposals of [29] which aims at the improvement of the convergence of the perturbative series in lattice PT.

The standard perturbative determination of the RC Z_O of a bilinear quark operator O_Γ (flavour indices are suppressed, unless necessary) requires the calculation of the quark self- energy and the amputated Green function $\Lambda_O(p)$ of (7) between external quark states, both at momentum p_μ (see, for example, [2]). Here we will work with the projected quark propagator $\Gamma_\mathcal{S}(p)$, and projected amputated Green function $\Gamma_O(p)$, defined as

$$\begin{aligned} \Gamma_\Sigma(p) &= \frac{-i}{48} Tr\left[\gamma_\mu \frac{\partial \mathcal{S}(p)^{-1}}{\partial p_\mu}\right] \\ \Gamma_O(p) &= \frac{1}{12} Tr\left[P_O \Lambda_O(p)\right] \end{aligned} \tag{14}$$

where the trace is over spin and colour indices and P_O is the Dirac matrix which renders the tree-level value of $\Gamma_O(p)$ equal to unity (i.e. it projects out the nominal Dirac structure of the Green function $\Lambda_O(p)$):

$$\begin{aligned} P_S &= I \quad ; \quad P_P = \gamma_5 \\ P_V &= \frac{1}{4}\gamma_\mu \quad ; \quad P_A = \frac{1}{4}\gamma_5\gamma_\mu \end{aligned} \tag{15}$$

From (12) we see that $\Gamma_\Sigma(p)$ renormalizes like

$$\hat{\Gamma}_\Sigma(p) = \lim\left[Z_\psi^{-1}\Gamma_\Sigma(p)\right] \tag{16}$$

The renormalization of $\Gamma_O(p)$ is identical to that of $\Lambda_O(p)$; c.f. eq(13):

$$\hat{\Gamma}_O(p) = \lim\left[Z_\psi^{-1} Z_O \Lambda_O(p)\right] \tag{17}$$

In the last two equations, the limit corresponds to the removal of the UV cutoff (e.g. $a \to 0$ on the lattice). The choice of $\Gamma_\Sigma(p)$ and $\Gamma_O(p)$, being scalar quantities, simplifies the discussion. Projected Green's functions have been implemented in recent works (e.g. [22]). Another simplification consists in working with zero quark mass. This is

[3] Dimensionless RC's which are not finite have the general dependence $Z_O(g_0^2, am, a\mu)$. Provided that the renormalization scale μ is chosen to be greater then the quark masses ($\mu \gg m$), there can be no dependence on am (same argument as above). The requirement of a well defined chiral limit allows only regular terms in m/μ, which however have been assumed negligible. Thus, in this case the RC's have the form $Z_O(g_0^2, a\mu)$

justified, as we have shown in the previous section, by the lack of mass dependence of Z_O, in the limit $\mu \gg m$.

Upon passing from formal definitions of renormalized quantities (like (17)) to their specific evaluation, we need to implement a regularization. The standard practice in continuum calculations consists in choosing a Dimensional Regularization (DR) scheme, such as Naive Dimensional Regularization (NDR), 't Hooft-Veltman (HV) etc. The expression for Γ_O, evaluated at 1-loop PT, can be written in the general form:

$$\Gamma_O^{DR}(p, g_0, \epsilon) = \left[1 + \frac{g_0(\mu)^2}{(4\pi)^2}\left(\gamma_\Gamma \frac{1}{\hat{\epsilon}} - \gamma_\Gamma \ln(p/\mu)^2 + C_\Gamma^{DR}\right)\right] + \mathcal{O}(\epsilon) \tag{18}$$

where γ_Γ is an anomalous dimension and C_Γ^{DR} is a finite constant, which depends on the regularization scheme and the gauge chosen. The factor $1/\hat{\epsilon}$ in (18) is an abbreviation for

$$\frac{1}{\hat{\epsilon}} = \frac{1}{\epsilon} + \ln(4\pi) - \gamma_E \tag{19}$$

where $\epsilon = (4 - D)/2$ and γ_E stands for Euler's constant. In DR we work in D dimensions, where the original bare coupling g_0 has dimension ϵ. Here the scale μ is introduced to render the bare coupling $g_0(\mu)$ dimensionless. The μ-dependence of the r.h.s. of (18) is apparent.

In order to renormalize Γ_O, we are in principle free to impose any renormalization scheme. The $\overline{\rm MS}$ scheme, at 1-loop, amounts to removing the $1/\hat{\epsilon}$ divergence. Since the RC of the projected Green function Γ_O is given by $Z_\Gamma = Z_\psi^{-1} Z_O$ (see (17)), this implies for Z_Γ the value

$$Z_\Gamma^{\overline{\rm MS},DR}(g_0(\mu), \epsilon) = 1 - \frac{g_0^2(\mu)}{(4\pi)^2}\gamma_\Gamma \frac{1}{\hat{\epsilon}} \tag{20}$$

Consequently, the renormalized Green function is given by

$$\begin{aligned}\hat{\Gamma}_O^{\overline{\rm MS}}(p, g_{\overline{\rm MS}}(\mu), \mu) &= \lim_{\epsilon \to 0}\left[Z_\Gamma^{\overline{\rm MS},DR}(g_0(\mu), \epsilon)\Gamma_O^{DR}(p, g_0, \epsilon)\right] \\ &= \left[1 + \frac{g_{\overline{\rm MS}}^2(\mu)}{(4\pi)^2}\left(-\gamma_\Gamma \ln(p/\mu)^2 + C_\Gamma^{DR}\right)\right]\end{aligned} \tag{21}$$

where, to this order, we are free to replace g_0 by $g_{\overline{\rm MS}}(\mu)$, the $\overline{\rm MS}$ renormalized coupling constant.

The same calculation can be repeated on the lattice. Now the UV cutoff is provided by the inverse finite lattice spacing a^{-1} and thus the 1-loop calculation yields:

$$\Gamma_O^{LAT}(p, g_0(a), a) = \left[1 + \frac{g_0(a)^2}{(4\pi)^2}(-\gamma_\Gamma \ln(pa)^2 + C_\Gamma^{LAT})\right] + \mathcal{O}(a) \tag{22}$$

where g_0 is the bare coupling of the lattice action. Note that at 1-loop the anomalous dimension γ_Γ is regularization independent.

The renormalization scheme can again be chosen at will; often the $\overline{\rm MS}$ is chosen also on the lattice. This seemingly unnatural choice (the $\overline{\rm MS}$ is closely linked to continuum DR) has a few advantages. For example, ME's of effective Hamiltonians, once calculated non-perturbatively on the lattice, must be renormalized and combined with perturbatively calculated Wilson coefficients, in order to obtain physical amplitudes. The renormalization-group invariance of these amplitudes is guaranteed only if the Wilson coefficients and the RC's are calculated in the same renormalization scheme. Since the former are often known in the $\overline{\rm MS}$ scheme, this scheme is also preferred for the calculation of the latter. This choice then corresponds to the requirement

$$\begin{aligned}\hat{\Gamma}_O^{\overline{\rm MS}}(p, g_{\overline{\rm MS}}(\mu), \mu) = \lim_{a \to 0}\Big[&Z_\Gamma^{\overline{\rm MS},LAT}(\mu a, g_0(a)) \\ &\times \Gamma_O^{LAT}(p, g_0(a), a)\Big]\end{aligned} \tag{23}$$

where again the lattice coupling $g_0(a)$ should be traded for the $\overline{\rm MS}$ renormalized coupling constant $g_{\overline{\rm MS}}(\mu)$. This point of principle is of limited relevance for a 1-loop calculation. From (21), (22) and (23), the following RC is obtained:

$$\begin{aligned}Z_\Gamma^{\overline{\rm MS},LAT}(\mu a, g_0(a)) = 1 + \frac{g_0(a)^2}{(4\pi)^2}\big[&\gamma_\Gamma \ln(\mu a)^2 \\ &+ C_\Gamma^{DR} - C_\Gamma^{LAT}\big]\end{aligned} \tag{24}$$

We now recall that the RC of the amputated vertex Γ_O is $Z_\Gamma = Z_\psi^{-1} Z_O$. The RC of the quark field Z_ψ can be calculated, from the renormalization of the quark propagator $\mathcal{S}(p)$, with an analogous procedure (c.f. (16)). The result is

$$\begin{aligned}Z_\psi^{\overline{\rm MS},LAT}(\mu a, g_0(a)) = 1 + \frac{g_0(a)^2}{(4\pi)^2}\big[&\gamma_\Sigma \ln(\mu a)^2 \\ &+ C_\Sigma^{DR} - C_\Sigma^{LAT}\big]\end{aligned} \tag{25}$$

Combining (24) and (25) we obtain

$$\begin{aligned}Z_O^{\overline{\rm MS},LAT}(\mu a, g_0(a)) = 1 + \frac{g_0(a)^2}{(4\pi)^2}[&\gamma_O \ln(\mu a)^2 \\ &+ \Delta_\Gamma + \Delta_\Sigma]\end{aligned} \tag{26}$$

where

$$\begin{aligned}\gamma_O &= \gamma_\Gamma + \gamma_\Sigma \\ \Delta_\Gamma &= C_\Gamma^{DR} - C_\Gamma^{LAT} \\ \Delta_\Sigma &= C_\Sigma^{DR} - C_\Sigma^{LAT}\end{aligned} \tag{27}$$

It is this RC (with this choice of renormalization condition) which is usually denoted by Z_O in lattice PT calculations. The dependence of $Z_O^{\overline{\rm MS},LAT}$ on the coefficients C_Γ^{DR} and C_Σ^{DR} comes from the choice of the $\overline{\rm MS}$ renormalization condition (see (21) and (23)) whereas its dependence on C_Γ^{LAT} and C_Σ^{LAT} from the lattice regularization (see (22)). Two perturbative calculations are thus necessary, one in the continuum for the C^{DR}'s and one on the lattice for C^{LAT}'s. The presence of the C^{DR}'s on the r.h.s. of (24) is sometimes called regularization dependence of the renormalization scheme [42,43]. The reader is referred to [2] for the explicit calculation of the RC's of several bilinear operators with the Wilson action. The corresponding results for the Clover action appeared in [7,8].

The PT method is applicable to finite and logarithmically divergent operators, the renormalization of which does not require a subtraction of lower dimensional operators. Model examples of operators with logarithmically divergent multiplicative RC's are the scalar density $S(x) = \bar{\psi}_1(x)\psi_2(x)$ and the pseudoscalar density $P(x) = \bar{\psi}_1(x)\gamma_5\psi_2(x)$. These are the cases best suited to our general discussion so far.

The RC's of the vector current $V^a_\mu(x)$ and the axial current $A^a_\mu(x)$ are also calculable in PT on similar lines. Being (partially) conserved quantities however, they have some peculiarities. Their (partial) conservation guarantees zero anomalous dimension (i.e $\gamma_V = 0$ and $\gamma_A = 0$). This result is regularization independent (e.g. valid both in the continuum (DR) and the lattice (LAT). It implies that the current renormalization constants are finite, scale independent quantities. Moreover, the following useful properties must be taken into consideration:

- The vector current conservation survives the continuum DR. Thus it satisfies a WI which implies that $Z_V^{\overline{\rm MS},DR} = 1$. This is an exact result, valid at all orders of PT. Therefore, at 1-loop $V^a_\mu(x)$ has zero finite constant; i.e. $C_V^{DR} + C_\Sigma^{DR} = 0$.
- As we will see in the next section, the local lattice vector current is not a conserved quantity. It has a non-vanishing 1-loop contribution (i.e. $C_V^{LAT} + C_\Sigma^{LAT} \neq 0$). In other words, $Z_V^{\overline{\rm MS},LAT}(g_0^2) \neq 1$.
- We will also show in the next section that on the lattice one can define an extended conserved vector current, $\widetilde{V}^a_\mu(x)$, which is also trivially renormalized with $Z_{\widetilde{V}}^{\overline{\rm MS},LAT} = 1$. Thus, also for this current we have $C_{\widetilde{V}}^{LAT} + C_\Sigma^{LAT} = 0$.
- For the axial current the argument is almost the same. However, some further peculiarities arise due to the breaking of the chiral symmetry by the regularization. In the continuum, the finite 1-loop contribution to the renormalization constant may or may not vanish, depending on the detailed choice of the regularization scheme. Thus, for NDR we have $C_A^{NDR} + C_\Sigma^{NDR} = 0$, implying $Z_A^{\overline{\rm MS},NDR} = 1$. On the other hand, with HV for example, we have $C_A^{HV} + C_\Sigma^{HV} \neq 0$, implying $Z_A^{\overline{\rm MS},HV}(g_0^2) \neq 1$. In the present work, we will always imply NDR when referring to DR.
- We will show in the next section that the lattice regularization breaks chiral symmetry. Thus on the lattice $C_A^{LAT} + C_\Sigma^{LAT} \neq 0$ (for any definition of the bare lattice current, local or extended). In other words, $Z_A^{\overline{\rm MS},LAT}(g_0^2) \neq 1$

We are also interested in the finite ratio Z_S/Z_P. Its finiteness relies on WI's (see next section) and implies that $\gamma_S = \gamma_P$. With the specific choice of NDR as the continuum dimensional regularization, we also have $C_S^{NDR} = C_P^{NDR}$.

3.1 Boosted improvement of the PT estimates of RC's

The perturbative calculation outlined above is only approximate, because of the truncation of the perturbative series. At present, most lattice RC's are only known in PT at 1-loop. Thus, they suffer from $\mathcal{O}(g_0^4)$ systematic errors. These errors must be born in mind when comparing the RC's, calculated in PT, to their non-perturbative estimates. In this subsection we discuss these systematic errors and comment on the proposals for improving the convergence of the perturbative series of [29]. These proposals are known as Boosted Perturbation Theory (BPT).

We have already pointed out that, in principle, at 1-loop there is an ambiguity in the choice of couplings ($g_0(a)$, $g_0(\mu)$, $g_{\overline{\rm MS}}(\mu)$) to be used in (24). This ambiguity is lifted in (and up to) 2-loop PT. At 1-loop however, a judicious choice of expansion parameter may be important in practice. The lattice coupling $g_0^2(a)$ has been shown to be a bad expansion parameter in [29], where several improved (boosted) couplings have been proposed, and several observables have been calculated in PT with these couplings.

In the specific case of Z_O, the Standard Perturbation Theory (SPT) result is given as a series in the bare (lattice) coupling g_0

$$Z_O(a\mu, g_0^2(a)) = 1 + C_1(a\mu)g_0^2(a) + C_2(a\mu)g_0^4(a) + \dots \quad (28)$$

To first order, the above equation reduces to (26), with $C_1(a\mu) = [\gamma_O \ln(a\mu)^2 + \Delta_\Gamma + \Delta_\Sigma]/(4\pi)^2$. According to [29], perturbative series have a better behaviour if expressed in terms of a renormalized coupling $g_V^2(q^*)$ at a suitable scale q^* (to be specified shortly):

$$Z_O(a\mu, g_V^2(q^*)) = 1 + C_1(a\mu)g_V^2(q^*) + \widetilde{C}_2(a\mu, aq^*)g_V^4(q^*) + \dots \quad (29)$$

The claim is that the perturbative estimate obtained from the boosted series, truncated at low orders, is closer to the non-perturbative result than its SPT counterpart.

The renormalized coupling $g_V(q)$ in (29) can be chosen at will. In [29] it is suggested to define it from the heavy quark potential $V(q)$ at momentum transfer q:

$$V(q) = -\frac{4g_V^2(q)}{3q^2} \quad (30)$$

Moreover, according to [29] and [44], the scale q^*, which is most appropriate to the expansion (29) of Z_O, is fixed by:

$$\ln(q^{*2}) = \frac{\int d^4q f(q, a\mu) \ln(q^2)}{\int d^4q f(q, a\mu)} \quad (31)$$

where $f(q, a\mu)$ is the function entering in the loop integral which defines the constant $C_1(a\mu)$ of (29): $C_1(a\mu) = \int d^4q f(q, a\mu)$.[4] From the above expression we readily see

[4] Here we are dealing with multiplicative renormalization, for which the above recipe is fairly non ambiguous. However, if the operator mixes with others under renormalization, the recipe is ambiguous, since one can either evaluate all mixing constants at the same scale q^* or obtain each mixing constant at a separate scale

that in general q^* depends on the renormalization scale μ. Here, however, we are only interested in the finite quantities Z_V, Z_A and Z_S/Z_P, for which the C_i's of (28) are μ-independent. Consequently, also q^* does not depend on μ.

The connection between the bare coupling g_0^2 and the renormalized coupling g_V^2 defined in (30) is obtained by calculating the heavy quark potential in lattice PT. At lowest order one finds

$$g_V^2(q) = g_0^2(a)[1 - g_0^2(a)2\beta_0 \ln(aq/c) + \mathcal{O}(g_0^4(a))] \quad (32)$$

where $\beta_0 = 11/(4\pi)^2$ and c is given by:

$$c = \pi \exp\left(\frac{4.702}{8\pi\beta_0}\right) \quad (33)$$

There are several possibilities, suggested in [29], for extracting $g_V^2(q^*)$ from (32). They can be classified as follows:

1. Purely Perturbative Boosting [PPB]: This method consists in using (32) in order to identify the couplings at scale $q = c/a$:

$$g_V^2(c/a) = g_0^2(a) + \mathcal{O}(g_0^4) \quad (34)$$

Then one can perform 2-loop Renormalization Group (RG) running of $g_V^2(c/a)$ down to the scale q^* in the standard fashion:

$$\frac{1}{g_V^2(q^*)} = \frac{1}{g_V^2(c/a)} + 2\beta_0 \ln(\frac{aq^*}{c}) + \frac{\beta_1}{\beta_0} \ln\left(\frac{g_V^2(c/a)}{g_V^2(q^*)}\right) \quad (35)$$

with $\beta_1 = 102/(4\pi)^4$. Thus, having calculated q^* (from (31)) and $g_V^2(q^*)$, we derive Z_O from (29). We call this estimate PPB.

2. Monte Carlo-Perturbative Boosting [MC-PB($\ln \Box$)]: Again this method is based on the relationship between g_V^2 and g_0^2 of (32). But now the boosted coupling constant is extracted from the non-perturbative (Monte Carlo) value of a short distance quantity, such as the average plaquette [45]. The logarithm of the plaquette, known as a perturbative series in g_0^2, is expressed, with the aid of (32), as a series in g_V^2. The leading coefficient of this series is an integral of the kind $\int d^4q f_\Box(q)$, which gives rise, through the criterion of (31), to the scale $q_\Box^*$, appropriate for this expansion. From [29] we have $q_\Box^* = 3.41/a$ and the expansion:

$$\ln(\frac{1}{3}Tr\langle U_\Box\rangle) = -\frac{1}{3}g_V^2(q_\Box^*)[1 - 9.46 \cdot 10^{-2} g_V^2(q_\Box^*) + \mathcal{O}(g_V^4)] \quad (36)$$

The series of the plaquette's logarithm has been preferred to that of the plaquette itself because, as claimed in [29], its perturbative expansion is better behaving. Using the Monte Carlo result for the average plaquette, (36) is solved for $g_V^2(q_\Box^*)$. 2-loops RG running subsequently yields the desired $g_V^2(q^*)$. With this value for $g_V^2(q^*)$, we derive Z_O from (29). We call this method MC-PB($\ln \Box$).

3. Monte Carlo-Perturbative Boosting [MC-PB($\Box$)]: Yet another proposal of [29] consists in combining (32), at the scale $q = \pi/a$, and the perturbative expansion of the plaquette

$$\frac{1}{3}Tr\langle U_\Box\rangle = 1 - \frac{1}{3}g_V^2(\pi/a) + 8.71 \cdot 10^{-2} g_V^4(\pi/a) + \mathcal{O}(g_V^6) \quad (37)$$

to obtain:

$$g_V^2(\pi/a)[1 - 0.513\frac{1}{4\pi}g_V^2(\pi/a) + \mathcal{O}(g_V^4)] = \frac{g_0^2(a)}{\frac{1}{3}Tr\langle U_\Box\rangle} \quad (38)$$

Once $g_V^2(\pi/a)$ is obtained from the above expression, two loop RG running yields $g_V^2(q^*)$ and subsequently Z_O. We call this estimate MC-PB($\Box$).

4. Naive Boosted Perturbation Theory [NBPT]: A particularly simple choice of boosted coupling [45] is $\tilde{g}^2 = g_0^2/(\frac{1}{3}Tr\langle U_\Box\rangle)$ and the subsequent substitution of g_0 by $\tilde{g}^2$ in (24). This recipe (a simplification of (38)) is denoted by NBPT. It has been implemented, for example, in [46] and [47]. As argued in [29], Mean Field arguments can also be used in order to support this prescription.

Having classified the various boostings, we now comment on them. All these methods should be in principle equivalent to lowest order in PT. From the field theoretic point of view, PPB is a perfectly legitimate operation, as it amounts to a different choice of expansion parameter. The two MC-PB prescriptions may be even superior to PPB in practice, since they possibly incorporate some non-perturbative and higher order effects. In order to implement these recipes to the perturbative evaluation of the RC's of interest, we have calculated the appropriate scale q^*; this is different for each RC. The values of q^* for Z_V, Z_A and Z_S/Z_P are collected in Table 1. Some technical details concerning their evaluation are gathered in the Appendix.

In Table 2, we collect the values of Z_O, obtained from the various boosting recipes listed above. We see that the variation of the boosted coupling, due to the choice of boosting recipe, does not exceed 20%, and that of the various RC's is about 10%. These variations reflect the systematic error introduced by the truncation of the perturbative series. In the next section, we will compare boosted results to the ones obtained non perturbatively from WI's. For this comparison we will use the MC-PB($\ln \Box$) prescription which is considered the optimal choice by the authors of [29]. As will be shown in Table 3, such a comparison reveals a general trend for BPT to reduce the magnitude of the $\mathcal{O}(g_0^4)$ systematic error. However, in some cases this error remains appreciable (more than 10%).

4 Lattice WI's with Wilson fermions

In this section we review the WI's which follow from the chiral flavour symmetry of the QCD action. They allow to compute the finite RC's of lattice operators in a non-perturbative way. Several properties of these RC's can be

Table 1. The values of the scale aq^*, characteristic of the boosted perturbative evaluation of Z_V, Z_A and Z_S/Z_P. The scales are shown for both the Wilson and Clover actions

Wilson			Clover		
$aq^*(Z_V)$	$aq^*(Z_A)$	$aq^*(Z_S/Z_P)$	$aq^*(Z_V)$	$aq^*(Z_A)$	$aq^*(Z_S/Z_P)$
2.4	2.6	1.9	2.7	1.2	3.2

Table 2. Wilson and Clover action RC's obtained with different boosting procedures (see text for notation). The RC's are shown at $\beta = 6.0$

Wilson - β=6.0						
	g^2	Z_V	g^2	Z_A	g^2	Z_S/Z_P
SPT	1.00	0.83	1.00	0.87	1.00	1.10
PPB	1.81	0.69	1.77	0.76	1.93	1.25
MC-PB(ln□)	2.14	0.63	2.08	0.72	2.32	1.34
MC-PB(□)	1.97	0.66	1.92	0.74	2.13	1.29
NBPT	1.68	0.71	1.68	0.77	1.68	1.20
Clover - β=6.0						
	g^2	Z_V	g^2	Z_A	g^2	Z_S/Z_P
SPT	1.00	0.90	1.00	0.98	1.00	1.20
PPB	1.75	0.83	2.25	0.96	1.67	1.42
MC-PB(ln□)	2.05	0.80	2.81	0.95	1.95	1.54
MC-PB(□)	1.90	0.81	2.52	0.96	1.81	1.48
NBPT	1.68	0.83	1.68	0.97	1.68	1.43

derived from the WI's. Although the results presented here are well known (see [48,49]), we give particular emphasis to the conditions (e.g. continuum limit, chiral limit etc.) under which these results are exact. This is essential to the understanding of the sources of systematic error in practical applications of the WI's, which form the basis of our analysis. The reader familiar with [48,49] may skip this section.

We consider local $SU_L(N_f) \times SU_R(N_f)$ chiral transformations of the fermionic fields:

$$\delta\psi(x) = i\left[\alpha_V^a(x)\frac{\lambda^a}{2} + \alpha_A^a(x)\frac{\lambda^a}{2}\gamma_5\right]\psi(x)$$
$$\delta\bar{\psi}(x) = -i\bar{\psi}(x)\left[\alpha_V^a(x)\frac{\lambda^a}{2} - \alpha_A^a(x)\frac{\lambda^a}{2}\gamma_5\right] \quad (39)$$

and examine the WI's derived separately in the vector and axial cases.

4.1 Vector WI's on the lattice

With degenerate quark masses (i.e. M_0 in (9) proportional to the unit matrix) global vector transformations are a symmetry of the action. From the corresponding local transformations ((39) with $\alpha_A^a = 0$) the following vector WI can be derived [48]:

$$i\langle\frac{\delta O(x_1,\ldots,x_n)}{\delta\alpha_V^a(x)}\rangle = a^4\sum_\mu \nabla_x^\mu\langle O(x_1,\ldots,x_n)\widetilde{V}_\mu^a(x)\rangle$$
$$+a^4\langle O(x_1,\ldots,x_n)\bar{\psi}(x)[\frac{\lambda^a}{2}, M_0]\psi(x)\rangle \quad (40)$$

where $O(x_1,\ldots,x_n)$ is any multi-local operator consisting of quark and gluon fields at different space-time points ($x_1 \neq x_2 \neq \ldots x_n$). $a\nabla_x^\mu f(x) = (f(x) - f(x-\mu))$ is an asymmetric lattice derivative, and

$$\widetilde{V}_\mu^a(x) = \frac{1}{2}[\bar{\psi}(x)(\gamma_\mu - 1)U_\mu(x)\frac{\lambda^a}{2}\psi(x+\mu)$$
$$+\bar{\psi}(x+\mu)(\gamma_\mu + 1)U_\mu^\dagger(x)\frac{\lambda^a}{2}\psi(x)] \quad (41)$$

is a point-split vector current. By keeping the point x separate from the points $x_1,\ldots,x_n$ and by taking the limit of degenerate bare quark masses, we see that (40) leads to the conservation of the point split lattice vector current, $\nabla_x^\mu\widetilde{V}_\mu^a(x) = 0$. This also implies that the standard local vector current $V_\mu^a(x) = \bar{\psi}(x)\frac{\lambda}{2}\gamma_\mu\psi(x)$ is not conserved on the lattice.

For the specific choice $O(x_1,x_2) = \psi(x_1)\bar{\psi}(x_2)$, the vector WI (40) becomes

$$\sum_\mu \nabla_x^\mu\langle\psi(x_1)\widetilde{V}_\mu^a(x)\bar{\psi}(x_2)\rangle$$
$$= \langle\psi(x_1)\bar{\psi}(x)[M_0, \frac{\lambda^a}{2}]\psi(x)\bar{\psi}(x_2)\rangle$$
$$+\delta(x_2-x)\langle\psi(x_1)\bar{\psi}(x_2)\rangle\frac{\lambda^a}{2}$$
$$-\delta(x_1-x)\frac{\lambda^a}{2}\langle\psi(x_1)\bar{\psi}(x_2)\rangle \quad (42)$$

Let us consider the case of degenerate quark masses ($M_0 \propto 1$) and place the vector current at the origin ($x = 0$). By Fourier-transforming the above equation and amputating the corresponding Green's function $G_{\widetilde{V}}^{\mu,a} = \langle\psi(x_1)\,\widetilde{V}^{\mu,a}(0)\,\bar{\psi}(x_2)\rangle$ (c.f. (5), (6) and (7)), we obtain

$$\sum_\mu\left[\frac{1-\exp(-iaq_\mu)}{a}\right]\Lambda_{\widetilde{V}}^{\mu,a}\left(p+\frac{q}{2}, p-\frac{q}{2}\right)$$
$$= -\mathcal{S}\left(p+\frac{q}{2}\right)^{-1}\frac{\lambda^a}{2} + \frac{\lambda^a}{2}\mathcal{S}\left(p-\frac{q}{2}\right)^{-1} \quad (43)$$

This is the lattice version of the vector WI. In the continuum limit, it is a formal expression relating bare quantities which are divergent. In order to render them meaningful, it is essential to go over to renormalized expressions. With the RC's strictly defined in the limit of vanishing cutoff, we express the bare quantities of (43) in terms of their renormalized counterparts, given in (12) and (13). Thus,

in the limit $a \to 0$, we obtain:

$$\sum_\mu i q_\mu Z_{\widetilde{V}}^{-1} \hat{\Lambda}_V^{\mu,a}\left(p+\frac{q}{2}, p-\frac{q}{2}\right) = -\hat{\mathcal{S}}\left(p+\frac{q}{2}\right)^{-1}\frac{\lambda^a}{2} + \frac{\lambda^a}{2}\hat{\mathcal{S}}\left(p-\frac{q}{2}\right)^{-1} \quad (44)$$

However, the definitions of the RC's of (12) and (13), used to derive the above result, are only formal. A concrete definition implies the choice of a renormalization scheme. In the case of the vector current, it is crucial to impose that the renormalized operator $\hat{V}_\mu^a$ satisfy the nominal WI of the continuum theory. From this requirement on (44) we obtain that the RC of the point split conserved vector current (41) is [48,49]:

$$Z_{\widetilde{V}} = 1 \quad (45)$$

This result, which is exact in the limit $a \to 0$, guarantees a proper definition of the vector charge and the validity of current algebra. Thus, the vector symmetry of the theory survives renormalization.

The local vector current $V_\mu^a(x) = \bar{\psi}(x)\frac{\lambda^a}{2}\gamma_\mu\psi(x)$ is not conserved on the lattice. However, it can be shown that its ME's differ from those of the conserved current by finite contributions. The argument goes as follows: first we express the the conserved current as

$$\widetilde{V}_\mu^a(x) = V_\mu^a(x) + \frac{a}{2}[\bar{\psi}(x)(\gamma_\mu - 1)\frac{\lambda^a}{2}\overrightarrow{\mathcal{D}}_\mu\,\psi(x) + \bar{\psi}(x)\overleftarrow{\mathcal{D}}_\mu\,(\gamma_\mu + 1)\frac{\lambda^a}{2}\psi(x)] \quad (46)$$

where we have used the lattice asymmetric covariant derivatives

$$a\overrightarrow{\mathcal{D}}_\mu\,\psi(x) = U_\mu(x)\psi(x+\mu) - \psi(x)$$
$$a\bar{\psi}(x)\overleftarrow{\mathcal{D}}_\mu = \bar{\psi}(x+\mu)U_\mu^\dagger(x) - \bar{\psi}(x) \quad (47)$$

The second term on the r.h.s. of (46) is a dimension-4 operator (we call it Δ_μ^a) multiplied by the lattice spacing. For definitiveness, we now consider the Green's function $\Lambda_{\widetilde{V}}(p)$ of (7). From (46) it follows that

$$\Lambda_{\widetilde{V}}(p) = \Lambda_V(p) + a\Lambda_\Delta(p) \quad (48)$$

The term $a\Lambda_\Delta(p)$ vanishes at tree-level in the naive continuum limit. Beyond tree-level, however, this term contributes, due to power divergences induced by the mixing with lower dimensional operators. The mixing with operators of equal dimension gives at most logarithmic divergences $\ln(ap)$; thus in the continuum limit such contributions vanish like $a\ln(ap)$. As far as lower dimensional operators are concerned, Δ_μ^a only mixes with the dimension-3 vector current V_μ^a. This gives rise to power divergences a^{-1}, without logarithms[5]. Thus, the term $a\Lambda_\Delta$ gives finite contributions, which combine with the renormalization of Λ_V to give the known non-renormalization exact result $Z_{\widetilde{V}} = 1$ of the l.h.s. of (48). Consequently, the term Λ_V on the r.h.s. has a finite renormalization:

$$Z_V(g_0^2) \neq 1 \quad (49)$$

Being finite, Z_V can only depend on the coupling g_0^2. Note that (44) (and any other vector WI) can now be expressed in terms of the non-conserved local current; we simply substitute $\hat{\Lambda}_V$ by $Z_V\Lambda_V$. This is the basis of the non-perturbative WI calculation of Z_V, detailed in Sect. 5.

One can also construct useful WI's based on hadronic correlation functions. For definitiveness, we consider the case in which the operator $O(x_1,\ldots,x_n)$ of (40) is $O(x_1,x_2) = P^{12}(x_1)P^{31}(x_2)$, where $P^{12} = \bar{\psi}_1\gamma_5\psi_2$. With $x \neq x_1, x_2$, so that contact terms do not contribute, (40) becomes

$$\sum_\mu \nabla_\mu^x \langle P^{12}(x_1)\widetilde{V}_\mu^{23}(x)P^{31}(x_2)\rangle = (m_2 - m_3)\langle P^{12}(x_1)S^{23}(x)P^{31}(x_2)\rangle \quad (50)$$

In analogy to (44), we can use (10) and (11) in order to express the above identity in terms of the renormalized densities, current and masses. By requiring that the renormalized quantities obey the nominal vector WI, we then find that the product of the quark mass times the scalar density is renormalization group invariant. Thus, we obtain for the RC of the scalar density $S(x)$:

$$Z_S = Z_m^{-1} \quad (51)$$

4.2 Axial WI's on the lattice

Far less straightforward is the implementation of the axial symmetry with Wilson fermions, because of the presence of the chiral symmetry breaking Wilson term in the action. This topic has been dealt with in [48,49]. The basic idea is that, by imposing suitable renormalization conditions, PCAC is recovered in the continuum. The axial WI, obtained from (39) with $\alpha_V^a = 0$, is

$$i\langle\frac{\delta O(x_1,\ldots,x_n)}{\delta\alpha_A^a(x)}\rangle = a^4\sum_\mu \nabla_x^\mu\langle O(x_1,\ldots,x_n)\widetilde{A}_\mu^a(x)\rangle - a^4\langle O(x_1,\ldots,x_n)\bar{\psi}(x)\{\frac{\lambda^a}{2}, M_0\}\gamma_5\psi(x)\rangle - a^4\langle O(x_1,\ldots,x_n)X^a(x)\rangle \quad (52)$$

where $\widetilde{A}_\mu^a(x)$ is a bilinear point-split axial current given by

$$\widetilde{A}_\mu^a(x) = \frac{1}{2}[\bar{\psi}(x)(\gamma_\mu\gamma_5)U_\mu(x)\frac{\lambda^a}{2}\psi(x+\mu) + \bar{\psi}(x+\mu)(\gamma_\mu\gamma_5)U_\mu^\dagger(x)\frac{\lambda^a}{2}\psi(x)] \quad (53)$$

[5] The absence of logarithms in such mixing has been explicitly demonstrated in [50] for the axial current at all orders in PT; the situation is analogous for the vector current

The term X^a in the above WI is the variation of the Wilson term under axial transformations:

$$X^a(x) = -\frac{1}{2}a\left[\bar{\psi}(x)\frac{\lambda^a}{2}\gamma_5\overrightarrow{\mathcal{D}}^2\psi(x) + \bar{\psi}(x)\overleftarrow{\mathcal{D}}^2\frac{\lambda^a}{2}\gamma_5\psi(x)\right] \quad (54)$$

where

$$\begin{aligned} a^2\overrightarrow{\mathcal{D}}^2\psi(x) &= \sum_\mu [U_\mu(x)\psi(x+\mu) \\ &\quad + U_\mu^\dagger(x-\mu)\psi(x-\mu) - 2\psi(x)] \\ a^2\bar{\psi}(x)\overleftarrow{\mathcal{D}}^2 &= \sum_\mu [\bar{\psi}(x+\mu)U_\mu^\dagger(x) \\ &\quad + \bar{\psi}(x-\mu)U_\mu(x-\mu) - 2\bar{\psi}(x)] \end{aligned} \quad (55)$$

Unlike the vector current case, X^a cannot be cast in the form of a four-divergence.

X^a is a dimension-4 operator which, in the naive continuum limit vanishes, being of the form ($a\times$dimension-5 operator). However, it has divergent ME's beyond tree-level. Its mixing with operators of equal and lower dimensions, worked out in [49], can be expressed as follows:

$$\overline{X}^a(x) = X^a(x) + \bar{\psi}(x)\{\frac{\lambda^a}{2}, \overline{M}\}\gamma_5\psi(x) + (Z_{\widetilde{A}} - 1)\nabla_x^\mu\widetilde{A}_\mu^a(x) \quad (56)$$

where naive dimensional arguments tell us that the mixing constant $Z_{\widetilde{A}}(g_0^2, am)$ is finite, whereas $\overline{M}(g_0^2, M_0)$ diverges linearly like a^{-1}. Logarithmic divergences have been shown to be absent at all orders in PT [50]. The mixing constants $Z_{\widetilde{A}}(g_0^2, am)$ and $\overline{M}(g_0^2, M_0)$, and therefore $\overline{X}^a(x)$, are defined by requiring that the renormalized axial current satisfies the nominal (continuum) axial WI. This amounts to a renormalization condition which ensures that the renormalized theory has the proper chiral symmetry. For example, with $O(x_1, x_2) = \psi(x_1)\bar{\psi}(x_2)$, (52) becomes

$$\begin{aligned} &\nabla_x^\mu\langle\psi(x_1)Z_{\widetilde{A}}\widetilde{A}_\mu^a(x)\bar{\psi}(x_2)\rangle \\ &= \langle\psi(x_1)\bar{\psi}(x)\{\frac{\lambda^a}{2}, (M_0 - \overline{M})\}\gamma_5\psi(x)\bar{\psi}(x_2)\rangle \\ &+ \langle\psi(x_1)\overline{X}^a(x)\bar{\psi}(x_2)\rangle - \\ &- \delta(x - x_2)\langle\psi(x_1)\bar{\psi}(x_2)\gamma_5\rangle\frac{\lambda^a}{2} \\ &- \delta(x - x_1)\frac{\lambda^a}{2}\langle\gamma_5\psi(x_1)\bar{\psi}(x_2)\rangle \end{aligned} \quad (57)$$

where X^a has been replaced by $\overline{X}^a$. We now take the continuum limit of the above equation, with the aim of obtaining from it the nominal axial WI for the renormalized quantities. We immediately see that this identity can be recovered from (57) provided that the following renormalization condition is imposed on $\overline{X}^a$:

$$\lim_{a\to 0}\langle\psi(x_1)\overline{X}^a(x)\bar{\psi}(x_2)\rangle = 0 \quad (58)$$

Moreover, the chiral limit is to be defined as the value M_C of M_0 for which the difference $M_C - \overline{M}(g_0^2, M_C)$ vanishes. Thus, in the chiral limit, (57) becomes

$$\begin{aligned} &\nabla_x^\mu\langle\psi(x_1)Z_{\widetilde{A}}\widetilde{A}_\mu^a(x)\bar{\psi}(x_2)\rangle \\ &= -\delta(x - x_2)\langle\psi(x_1)\bar{\psi}(x_2)\gamma_5\rangle\frac{\lambda^a}{2} \\ &- \delta(x - x_1)\frac{\lambda^a}{2}\langle\gamma_5\psi(x_1)\bar{\psi}(x_2)\rangle \end{aligned} \quad (59)$$

Expressing the quark fields in terms of their renormalized counterparts, we immediately see that the nominal axial WI is recovered up to vanishing cutoff effects, provided that we interpret the product $\hat{A}_\mu^a = Z_{\widetilde{A}}\widetilde{A}_\mu^a$ as the renormalized axial current. Note that $Z_{\widetilde{A}}(g_0^2, am)$, having been fixed in the continuum limit by (58), does not depend on am (only a regular dependence on am permits a well defined chiral limit). Thus, it is a finite renormalization of the form $Z_{\widetilde{A}}(g_0^2)$.

The above arguments rest on the assumption that the chiral point M_C, defined after (58), is consistent with other definitions (e.g. the vanishing of the quark mass in PT or of the pion mass). This has been verified both in 1-loop PT [49] and non-perturbatively (e.g. [16]).

In practice it turns out to be more convenient to work with the lattice local axial current $A_\mu^a(x)$. We can show, in a fashion analogous to the case of the vector current (c.f. the power counting argument based on (46)–(48)), that $A_\mu^a(x)$ has a finite normalization constant Z_A. Thus we have $\hat{A}_\mu^a = \lim_{a\to 0}[Z_{\widetilde{A}}\widetilde{A}_\mu^a] = \lim_{a\to 0}[Z_A A_\mu^a]$. From now on, the combination $Z_{\widetilde{A}}\widetilde{A}_\mu^a$ will always be substituted by $Z_A A_\mu^a$ wherever it appears in a WI. Also analogous to the vector current case is the lack of mass dependence of these constants. We therefore have:

$$Z_{\widetilde{A}}(g_0^2)\ ,\ Z_A(g_0^2)\ \neq\ 1 \quad (60)$$

Hadronic axial WI's are also very useful. In (52) we consider the operator $O(x_1) = P^{21}(x_1) = \bar{\psi}_2(x_1)\gamma_5\psi_1(x_1)$ and $x \neq x_1$, so that contact terms do not contribute. The resulting WI is

$$\begin{aligned} Z_A\sum_\mu\nabla_x^\mu\langle A_\mu^{12}(x)P^{21}(x_1)\rangle &= \langle\overline{X}^{12}(x)P^{21}(x_1)\rangle + \\ &+ \big[m_{01} + m_{02} - \overline{m}_1 - \overline{m}_2\big]\langle P^{12}(x)P^{21}(x_1)\rangle \end{aligned} \quad (61)$$

where $\overline{m}_i(g_0^2, M_0)$ is the i^{th} diagonal element of $\overline{M}(g_0^2, M_0)$. Note that we have replaced X by $\overline{X}$ according to (56). The condition (58) implies that correlation function of the operator $\overline{X}$ vanishes in the continuum limit, except for localized contact terms. Thus, when the points x_1 and x are kept separate, $\langle\overline{X}(x)\hat{P}(x_1)\rangle$ vanishes for $a \to 0$. We now write in (61) the renormalized quark mass as:

$$\begin{aligned} m_R = \overline{Z}_m[m_0 - \overline{m}(m_0)] = \overline{Z}_m[m_0 - m_C - \left.\frac{\partial\overline{m}}{\partial m_0}\right|_{m_C} \\ \times(m_0 - m_C) + ...] \end{aligned} \quad (62)$$

where we have used the property $\overline{m}(m_C) = m_C$. Upon expressing (61) in terms of renormalized quantities and requesting the validity of the axial WI in the continuum limit, we obtain for the RC Z_P of the pseudoscalar density:

$$Z_P = 1/\overline{Z}_m \tag{63}$$

Note that from the WI of (61) we can only determine the combination

$$\begin{aligned} 2\rho^{12} &= Z_A^{-1}[m_{01} + m_{02} - \overline{m}_1 - \overline{m}_2] \\ &= \frac{\sum_\mu \nabla^x_\mu \langle A^{12}_\mu(x) P^{21}(x_1)\rangle}{\langle P^{12}(x) P^{21}(x_1)\rangle} \end{aligned} \tag{64}$$

but not Z_A. Again, the above is true up to vanishing cutoff effects.

A very useful WI can be derived taking $O(x_1, x_2) = A^b_\nu(x_1) V^c_\rho(x_2)$:

$$\begin{aligned} & Z_A \nabla^\mu_x \langle A^a_\mu(x) A^b_\nu(x_1) V^c_\rho(x_2)\rangle \\ &\quad = \langle \bar\psi(x) \{\frac{1}{2}\lambda^a, (M_0 - \overline{M})\}\gamma_5 \psi(x) A^b_\nu(x_1) V^c_\rho(x_2)\rangle \\ &\quad + \langle \overline{X}^a(x) A^b_\nu(x_1) V^c_\rho(x_2)\rangle + \\ &\quad + i f^{abd} \delta(x - x_1) \langle V^d_\nu(x_1) V^c_\rho(x_2)\rangle \\ &\quad + i f^{acd} \delta(x - x_2) \langle A^b_\nu(x_1) A^d_\rho(x_2)\rangle \end{aligned} \tag{65}$$

where again use of (56) has been made. By symmetry arguments the contact terms to which $\overline{X}^a$ gives rise must have the form

$$\begin{aligned} & \langle \overline{X}^a(x) A^b_\nu(x_1) V^c_\rho(x_2)\rangle \\ &\quad = -i c_1 f^{abd} \delta(x - x_1) \langle V^d_\nu(x_1) V^c_\rho(x_2)\rangle + \\ &\quad + i c_2 f^{acd} \delta(x - x_2) \langle A^b_\nu(x_1) A^d_\rho(x_2)\rangle + \dots \end{aligned} \tag{66}$$

where the dots stand for localized Schwinger terms, which will vanish after integration over x (we always keep $x_1 \neq x_2$). We now proceed as follows: (I) rewrite (65), using (66) and expressing all bare quantities in terms of the renormalized ones; (II) recall that the RC $\overline{Z}_m$ of $M_0 - \overline{M}$ is the inverse of the RC Z_P (c.f. (63)); (III) require, as always, that the renormalized quantities obey the nominal axial WI. Thus we obtain

$$\begin{aligned} c_1 &= 1 - \frac{Z_V}{Z_A} \\ c_2 &= \frac{Z_A}{Z_V} - 1 \end{aligned} \tag{67}$$

The above conditions ensure the recovery of the canonical WI in the continuum limit. This WI is the basis of the non-perturbative calculation of Z_A in the fashion of [11, 17], where (64)–(67) are combined into

$$\begin{aligned} & \langle [\nabla^\mu_x A^a_\mu(x) - \bar\psi(x) \{\frac{\lambda^a}{2}, \rho\}\gamma_5 \psi(x)] A^b_\nu(x_1) V^c_\rho(x_2)\rangle \\ &\quad = + i f^{abd} \delta(x - x_1) \langle V^d_\nu(x_1) V^c_\rho(x_2)\rangle \frac{Z_V}{Z_A^2} \\ &\quad + i f^{acd} \delta(x - x_2) \langle A^b_\nu(x_1) A^d_\rho(x_2)\rangle \frac{1}{Z_V} \end{aligned} \tag{68}$$

with $\rho = Z_A^{-1}(M_0 - \overline{M})$ a matrix generalization of (64). Performing the integration over x, and integrating over $\mathbf{x}_1$ (recall that $x_1 \neq x_2$ is necessary in order to eliminate Schwinger terms) we arrive at:

$$\begin{aligned} & \int dx \int d\mathbf{x}_1 \langle \bar\psi(x) \{\frac{1}{2}\lambda^a, \rho\}\gamma_5 \psi(x) A^b_\nu(x_1) V^c_\rho(x_2)\rangle = \\ &\quad - i \frac{Z_V}{Z_A^2} f^{abd} \int d\mathbf{x}_1 \langle V^d_\nu(x_1) V^c_\rho(x_2)\rangle \\ &\quad - i \frac{1}{Z_V} f^{acd} \int d\mathbf{x}_1 \langle A^b_\nu(x_1) A^d_\rho(x_2)\rangle \end{aligned} \tag{69}$$

Note that this equation is only valid away from the chiral limit, where the integral over x of the total divergence $\nabla^\mu_x A^a_\mu(x)$ vanishes. At zero quark mass, the term containing the total divergence contributes because of the presence of massless Goldstone bosons.

The same arguments may be repeated for the operator $O(x_1, x_2) = S^g(x_1) P^h(x_2)$. By imposing that the renormalized operators $\hat{S}^f(x_1)$ and $\hat{P}^f(x_2)$ satisfy the (integrated) nominal Ward identity, we arrive at

$$\begin{aligned} & \int d^4x \int d^3\mathbf{y}\ \langle \bar\psi(x) \{\frac{1}{2}\lambda^f, \rho\}\psi(x) S^g(x_1) P^h(x_2)\rangle \\ &\quad = \frac{Z_P}{Z_A Z_S} d^{fgl} \int d^3\mathbf{x}_1\ \langle P^l(x_1) P^h(x_2)\rangle \\ &\qquad + \frac{Z_S}{Z_A Z_P} d^{fhl} \int d^3\mathbf{x}_1\ \langle S^g(x_1) S^l(x_2)\rangle \end{aligned} \tag{70}$$

where the d's are defined in (4). The WI of (70) is valid if $f \neq g$ and $f \neq h$, otherwise there are additional terms containing flavour-singlet currents on the right hand side. Analogous arguments to the ones used previously, show that the ratio Z_S/Z_P, being obtained from a WI, is a finite function of g_0^2. Thus, the densities S and P have the same anomalous dimension.

Another way of obtaining the ratio Z_P/Z_S is found by combining (10), (51), (62) and (63):

$$\frac{Z_P}{Z_S} = \frac{m_0 - \overline{m}(m_0)}{m_0 - m_C} = 1 - \left.\frac{\partial \overline{m}(g_0^2, m_0)}{\partial m_0}\right|_{m_0 = m_C} + \dots \tag{71}$$

where $m_0 - \overline{m} = 2\rho Z_A$.

In this section we have reviewed the lattice vector and axial WI's, emphasizing that they explicitly determine finite RC's such as Z_V, Z_A and the ratio Z_S/Z_P. We have repeatedly stressed that these results are strictly true up to vanishing $\mathcal{O}(a)$ terms. In the next section we will use the above WI's in order to determine non-perturbatively the RC's. A crucial observation is that these computations are performed at fixed finite cutoff a. In this case, the continuum limit is never taken, and an apparent "dependence" of Z_O on a is observed. This "dependence" is a discretization (systematic) error, which will eventually drop out in the continuum limit. Minimizing this error is the focal point of the Clover action improvement, discussed in Subsect. 5.2.

5 Evaluation of RC's from WI's

The most accurate method for a non-perturbative determination of RC's is based on chiral WI's. Only finite RC's can be extracted from them, such as Z_V, Z_A and the ratio Z_S/Z_P. We will use the general results of the previous section in order to show how they can be computed non-perturbatively.

These RC's are determined by requiring that bare correlation functions, calculated numerically on the lattice at finite UV cutoff a^{-1}, when renormalized, obey the WI's of the continuum theory. This requirement, which we have justified in the previous section in the $a \to 0$ limit, should be also valid at very small but finite lattice spacing, within a given accuracy. At fixed lattice spacing, correlation functions can be computed numerically, providing us with an intrinsically non-perturbative estimate of each term of a WI. These WI's can then be solved for the several RC's. Clearly, this method is free from the errors which affect the perturbative estimates of the RC's (i.e. higher orders in g_0^2). It suffers however, from finite cutoff effects. We shall address this question in detail in the rest of this paper.

5.1 Wilson WI estimates of RC's

First we consider the WI computation of the RC of the vector current, from suitable ratios of correlation functions. We shall use the following symmetrized form of the conserved current:

$$V_\mu^{aC} = \frac{1}{2}[\widetilde{V}_\mu^a(x) + \widetilde{V}_\mu^a(x-\mu)] \tag{72}$$

Being just a symmetrization of $\widetilde{V}_\mu^a$, it has the same RC, $Z_{V^C} = 1$. From the discussion of the vector WI's between hadronic correlation functions, we conclude that Z_V can be derived from ratios of n-point correlation functions; e.g.

$$\begin{aligned}
R_\rho(\mathbf{q}) &= \frac{\int d\mathbf{x} e^{i\mathbf{qx}} \langle V_k^C(x) M_\rho^{k\dagger}(0)\rangle}{\int d\mathbf{x} e^{i\mathbf{qx}} \langle V_k(x) M_\rho^{k\dagger}(0)\rangle} \\
&\to \frac{\sum_r \epsilon_r^k(\mathbf{q}) \langle 0|V_k^C(0)|\rho_r(\mathbf{q})\rangle}{\sum_r \epsilon_r^k(\mathbf{q}) \langle 0|V_k(0)|\rho_r(\mathbf{q})\rangle} + \dots \qquad (73) \\
R_0(\mathbf{q}) &= \frac{\int d\mathbf{x} \int d\mathbf{y} e^{i\mathbf{qx}} \langle P_K(0) V_0^C(x) P_D^\dagger(y)\rangle}{\int d\mathbf{x} \int d\mathbf{y} e^{i\mathbf{qx}} \langle P_K(0) V_0(x) P_D^\dagger(y)\rangle} \\
&\to \frac{\langle K(\mathbf{q})|V_0^C(0)|D(\mathbf{0})\rangle}{\langle K(\mathbf{q})|V_0(0)|D(\mathbf{0})\rangle} + \dots \\
R_k(\mathbf{q}) &= \frac{\int d\mathbf{x} \int d\mathbf{y} e^{i\mathbf{qx}} \langle P_K(0) V_k^C(x) P_D^\dagger(y)\rangle}{\int d\mathbf{x} \int d\mathbf{y} e^{i\mathbf{qx}} \langle P_K(0) V_k(x) P_D^\dagger(y)\rangle} \\
&\to \frac{\langle K(\mathbf{q})|V_k^C(0)|D(\mathbf{0})\rangle}{\langle K(\mathbf{q})|V_k(0)|D(\mathbf{0})\rangle} + \dots \\
R_k^*(\mathbf{q}) &= \frac{\int d\mathbf{x} \int d\mathbf{y} e^{i\mathbf{qx}} \langle M_{K^*}^\lambda(0) V_k^C(x) P_D^\dagger(y)\rangle}{\int d\mathbf{x} \int d\mathbf{y} e^{i\mathbf{qx}} \langle M_{K^*}^\lambda(0) V_k(x) P_D^\dagger(y)\rangle} \\
&\to \frac{\sum_r \epsilon_r^\lambda(\mathbf{q}) \langle K_r^*(\mathbf{q})|V_k^C(0)|D(\mathbf{0})\rangle}{\sum_r \epsilon_r^\lambda(\mathbf{q}) \langle K_r^*(\mathbf{q})|V_k(0)|D(\mathbf{0})\rangle} + \dots
\end{aligned}$$

where $k = 1, 2, 3$ is a spatial index and $\epsilon_r^\lambda(\mathbf{q})$ ($\lambda = 0, \dots, 3$) the vector meson polarization vectors. For definitiveness, we have specified the quark flavours by considering the correlations characteristic of the ρ decay (2-point functions) and those of the $D \to K$ and $D \to K^*$ decays (3-point functions). Thus, the operators are defined as: $P_K = \bar{u}\gamma^5 s$; $P_D = \bar{u}\gamma^5 c$; $M_\rho^k = \bar{u}\gamma^k d$; $M_{K^*}^\lambda = \bar{u}\gamma^\lambda s$. The vector currents have a $\bar{u} - d$ flavour structure in R_ρ and a $\bar{u} - c$ flavour structure in all other ratios. In the above equation, we also show the asymptotic behaviour of these ratios, obtained at large time separations, up to contributions from higher excited states. All these ratios give estimates of Z_V up to terms of $\mathcal{O}(a)$.

The above ratios represent the most straightforward way to obtain Z_V, but it is by no means the only one. As an example, we present a method, which follows more closely the vector WI's but is analogous to the determination of Z_V from the ratio R_0. It consists in integrating over all 3-space $\mathbf{x}$ both sides of (50). Upon integrating, the spatial derivative $\nabla^k \widetilde{V}_k$ vanishes (either because fields die-off at infinity or due to periodic boundary conditions of a finite lattice). Then we write the WI in terms of V_0^C. The result is

$$\begin{aligned}
&\int d\mathbf{x} \langle P^{12}(x_1) \nabla_x^0 V_0^C(x) P^{31}(x_2)\rangle = \frac{1}{2}(m_2 - m_3) \\
&\times \int d\mathbf{x} \langle P^{12}(x_1)[S^{23}(x) + S^{23}(x - \hat{0})] P^{31}(x_2)\rangle \qquad (74)
\end{aligned}$$

where, upon passing from $\widetilde{V}_0$ to V_0^C, we end up with the sum of two adjacent scalar densities on the r.h.s. Finally, we substitute $Z_V V_0$ for V_0^C in the above and solve for Z_V. The implementation of (73) for the computation of Z_V will be referred to as the "Ratio" determination. When WI's are directly used instead, we will call it the WI determination. Both are non-perturbative methods, equivalent in the $a \to 0$ limit.

For the WI determination of Z_A we start from (69). For definitiveness, we will consider 3 quark flavours, which may or may not be degenerate in mass. It is necessary to keep track of the flavour content of each operator, e.g. $O_\Gamma^{12}(x) = \bar{\psi}_1(x) \Gamma \psi_2(x)$. The RC's do not depend on flavour. We shall however leave flavour indices on them in order to remind us the origin of possible discretization errors in their determination (e.g. $\mathcal{O}(am_1, am_2)$ for Z^{12}). With this flavour content, setting $x_2 = 0$, (69) becomes

$$\begin{aligned}
2\rho^{12} &\int dx \int d\mathbf{x}_1 \langle P^{12}(x) A_\nu^{31}(x_1) V_\rho^{23}(0)\rangle = \\
&= \frac{Z_V^{23}}{Z_A^{13} Z_A^{12}} \int d\mathbf{x}_1 \langle V_\nu^{32}(x_1) V_\rho^{23}(0)\rangle \\
&- \frac{Z_A^{13}}{Z_A^{12} Z_V^{23}} \int d\mathbf{x}_1 \langle A_\nu^{31}(x_1) A_\rho^{13}(0)\rangle \qquad (75)
\end{aligned}$$

This WI can be useful in several ways. To simplify matters, in practice we will consider the case of flavours 2 and 3 degenerate in mass. This means that the pre-factors of the two integrals of the r.h.s. of (75) simplify to $Z_V^{22}/(Z_A^{12})^2$ and $1/Z_V^{22}$. Thus, by solving the above WI at different

times t_1, we obtain estimates of Z_V^{22} and Z_A^{12}. Alternatively, Z_V can be calculated from, say, one of the ratios of (73) and used in (75), from which Z_A can be computed. Moreover, with $\nu = \rho = 0$, the integrated correlation $\int d\mathbf{x}_1 \langle V_0(x_1) V_0(0)\rangle$ vanishes, since it is proportional to the charge of the vacuum. This means that the remaining WI

$$2\rho^{12} \int dx \int d\mathbf{x}_1 \langle P^{12}(x) A_0^{21}(x_1) V_0^{22}(0)\rangle = -\frac{1}{Z_V^{22}} \int d\mathbf{x}_1 \langle A_0^{21}(x_1) A_0^{12}(0)\rangle \tag{76}$$

can be used for yet another determination of Z_V. Besides Z_A and Z_V, the finite ratio Z_S/Z_P can also be obtained from the WI of (70), by a method analogous to the one used for Z_A with (75).

So far only WI's of hadronic correlation functions have been examined. Quark correlation functions such as $\langle \psi(x) V_\mu(0) \bar{\psi}(y)\rangle$ and $\langle \psi(x) A_\mu(0) \bar{\psi}(y)\rangle$ can also be used in principle with WI's such as (42) and (57). However, in numerical simulations the ratios of hadronic correlation functions turn out to be more stable than those of quark correlation functions. For the latter case, we must work in a fixed gauge and the Gribov ambiguity causes increased statistical fluctuations. Thus, the determination of Z_V and Z_A from quark state correlation functions is expected to be noisier. This has been explicitly verified for Z_A in [17] and [19].

5.2 Clover-improved WI estimates of RC's

As already pointed out, the WI determination of the finite RC's has the advantage of being non-perturbative. In principle, all WI's should give the same finite Z_O of a given operator O, since the RC's are characteristic of the operator they renormalize, but are independent of the ME's from which they are obtained. In practice, however, problems arise from the fact that the RC's, properly defined at vanishing cutoff, when calculated from WI's (or Ratios), are obtained at finite cutoff. At finite lattice spacing, operators mix with higher dimensional (irrelevant) operators. This mixing spoils the renormalization properties of the lattice operators, which is only recovered in the limit $a \to 0$. The contamination from these higher dimensional operators to the numerical calculation of RC's depends on the correlation functions (or ME's) used. Consequently, the result for Z_O obtained at finite cutoff will depend on the correlation functions from which it has been obtained and the quark mass at which the simulation was performed. This unwanted dependence signals the presence of systematic errors which are effectively $\mathcal{O}(a)$ in the scaling limit. More specifically, they are $\mathcal{O}(a\Lambda_{QCD}), \mathcal{O}(am), \mathcal{O}(aq)$ and higher, where m and q stand for the masses and spatial momenta that characterize the process under study.

In order to reduce this systematic error, several proposals have been put forward [51], [30], [31], [32]. All are based on the Symanzik improvement programme [52], which consists in adding suitable irrelevant dimension-5 operators to the lattice action. Moreover, d-dimensional operators must also be redefined by the addition of $(d+1)$-dimensional improving operators. All proposals cited above achieve the elimination of the $\mathcal{O}(a)$ discretization error (although at different orders in PT), and are referred to as $\mathcal{O}(a)$ improvement. The most widely used improved action is the SW/Clover one [30], defined as follows:

$$S_c = S_f - c_{SW}(g_0^2) a^4 \sum_{x,\mu\nu} \frac{a}{4} \bar{\psi}(x) \sigma_{\mu\nu} F_{\mu\nu}(x) \psi(x) \tag{77}$$

with $F_{\mu\nu}(x)$ the clover-leaf discretization of the field tensor. The improving coefficient $c_{SW}(g_0^2)$ can be calculated in PT [53]; more recently a non-perturbative determination has also been performed [32]. In this work we limit ourselves to its tree-level value $c_{SW} = 1$. At this order, all $\mathcal{O}(a g_0^{2n} \ln^n a)$ terms, which are effectively of $\mathcal{O}(a)$ in the scaling limit ($g_0^2 \sim 1/\ln a$), are eliminated from correlation functions. At leading-log level, the improvement of local operators can be expressed as a rotation of the fermion fields [31]. According to this prescription, improved bilinear operators are given by

$$O_\Gamma^I(x) = \bar{\psi}^R(x) \Gamma \psi^R(x) \tag{78}$$

where the rotated fields are defined through

$$\psi^R(x) = \left[1 - \frac{a}{4}\left(\vec{\not{D}} - m_0\right)\right] \psi(x)$$
$$\bar{\psi}^R(x) = \bar{\psi}(x) \left[1 + \frac{a}{4}\left(\overleftarrow{\not{D}} + m_0\right)\right] \tag{79}$$

and the symmetric lattice covariant derivatives are:

$$a \vec{D}_\mu \psi(x) = \frac{1}{2}[U_\mu(x)\psi(x+\hat{\mu}) - U_\mu^\dagger(x-\hat{\mu})\psi(x-\hat{\mu})]$$
$$a\bar{\psi}(x) \overleftarrow{D}_\mu = \frac{1}{2}[\bar{\psi}(x+\hat{\mu})U_\mu^\dagger(x) - \bar{\psi}(x-\hat{\mu})U_\mu(x-\hat{\mu})] \tag{80}$$

The improved operators O^I differ from the original ones by terms proportional to the cutoff. Consequently, they have different RC's, say Z_{O^I}.

For the Clover action, the quark propagator $\mathcal{S}^I(x-y) = \langle \psi(x) \bar{\psi}(y)\rangle$ satisfies the equations

$$\left[\vec{\not{D}}(x) + m_0 + a\vec{W}\right] \mathcal{S}^I(x-y) = \delta(x-y)$$
$$\mathcal{S}^I(x-y) \left[-\overleftarrow{\not{D}}(y) + m_0 + a\overleftarrow{W}\right] = \delta(x-y) \tag{81}$$

The terms $\vec{W}$ and $\overleftarrow{W}$ of (81) are shorthand notation for the operators arising from the variation with respect to ψ and $\bar{\psi}$ of the Wilson term of the fermionic action, including the Clover term given by (77). The tree-level $\mathcal{O}(a)$ improved quark propagator is $\langle \psi^R(x) \bar{\psi}^R(y)\rangle$. Using (79) and (81),

we can write it in terms of $\mathcal{S}^I(x-y)$:

$$\langle\psi^R(x)\bar{\psi}^R(y)\rangle = \left[1-\frac{a}{2}\overrightarrow{\not{D}}(x)\right]\mathcal{S}^I(x-y)\left[1+\frac{a}{2}\overleftarrow{\not{D}}(y)\right] + \frac{a}{2}\delta(x-y)+\mathcal{O}(a^2) \quad (82)$$

where $\mathcal{O}(a^2)$ terms proportional to $\overrightarrow{W}$ and $\overleftarrow{W}$ have been dropped, as they do not affect $\mathcal{O}(a)$ improvement. In computing correlation functions between external on-shell hadron states, the δ-function never contributes. Therefore these correlation functions can be expressed directly in terms of the effective rotated propagator

$$\mathcal{S}^{eff}(x-y) = \left[1-\frac{a}{2}\overrightarrow{\not{D}}(x)\right]\mathcal{S}^I(x-y) \times\left[1+\frac{a}{2}\overleftarrow{\not{D}}(y)\right] \quad (83)$$

This is the Clover propagator we will mostly use in our computations. The reason it contains an $\mathcal{O}(a^2)$ term in its definition is that it is readily computable in numerical simulations [16]. Thus, it has been extensively used in several Clover improved lattice QCD computations of on-shell ME's. However, upon studying the lattice WI's we find cases for which the space-time points of correlation functions are allowed to coincide. The on-shell ME's argument is then invalidated and the complete propagator of (82) must be used, with the δ-function giving rise to contact terms.

More generally, we can add to the improved propagator $\langle\psi^R(x)\bar{\psi}^R(y)\rangle$ the two equations of motion (81), multiplied by an arbitrary factor z. Up to $\mathcal{O}(a^2)$, we then find

$$\begin{aligned}\langle\psi^R(x)\bar{\psi}^R(y)\rangle_z = &\left[1-\frac{a}{2}\left(z\overrightarrow{\not{D}}(x)-(1-z)m_0\right)\right]\mathcal{S}^I(x-y)\\ &\times\left[1+\frac{a}{2}\left(\overleftarrow{\not{D}}(y)z+(1-z)m_0\right)\right]\\ &+\frac{a}{2}(2z-1)\delta(x-y)+\mathcal{O}(a^2)\end{aligned} \quad (84)$$

In this case, improved on-shell correlation functions can be expressed in terms of traces of the effective propagator

$$\mathcal{S}_z^{eff}(x-y) = \left[1-\frac{a}{2}\left(z\overrightarrow{\not{D}}(x)-(1-z)m_0\right)\right]\mathcal{S}^I(x-y) \times\left[1+\frac{a}{2}\left(\overleftarrow{\not{D}}(y)z+(1-z)m_0\right)\right] \quad (85)$$

with z a free parameter, which we have at our disposal for optimization purposes. Clearly, for different choices of z, these ME's (and the corresponding improved operators) differ by $\mathcal{O}(a^2)$ terms. This implies that their RC's also vary with z. In this work, unless otherwise stated, we shall use operators O_Γ^I which correspond to $z=1$.

Although on-shell correlation functions of local improved operators O_Γ^I are now improved up to $\mathcal{O}(g_0^2a)$, the conserved (point-split) current is not improved. An improved conserved current is given by [15]

$$V_\mu^{CI}(x) = V_\mu^C + \frac{1}{2}a\overline{\nabla}_\rho(\bar{\psi}\sigma_{\rho\mu}\psi) \quad (86)$$

where the extra "$\sigma_{\rho\mu}$ term" has a symmetric lattice derivative ($2a\overline{\nabla}_\mu f(x) = f(x+\mu)-f(x-\mu)$).

Having improved both local and conserved operators, we can now use them in the ratios of (73), in order to obtain estimates of Z_{V^I} which only suffer from $\mathcal{O}(ag_0^2)$ discretization errors. On the other hand, the l.h.s. of the WI of (74), even when expressed in terms of P^I and V_0^{CI}, still suffers from $\mathcal{O}(a)$ corrections. This is due to the presence of the asymmetric lattice derivative $\nabla_x^\mu = \partial_x^\mu + \mathcal{O}(a)$. In order to improve it, we must express it in terms of the symmetric derivative $\overline{\nabla}_x^\mu = \partial_x^\mu + \mathcal{O}(a^2)$ and make use of the fact that the conserved currents V_μ^C and V_μ^{CI} have the same divergence ($\overline{\nabla}^\mu V_\mu^C = \overline{\nabla}^\mu V_\mu^{CI}$). Thus we arrive at the following $\mathcal{O}(a)$ improved version of (74):

$$\begin{aligned}&\int d\mathbf{x}\langle P^{12}(x_1)\overline{\nabla}_x^0 V_0^{CI}(x)P^{31}(x_2)\rangle \\ &\quad= \frac{(m_2-m_3)}{4}\int d\mathbf{x}\langle P^{12}(x_1)[2S^{23}(x)\\ &\quad+S^{23}(x+\hat{0})+S^{23}(x-\hat{0})]P^{31}(x_2)\rangle\end{aligned} \quad (87)$$

In the above equation we have dropped the I superscript from the density P. The factor 1/4 on the r.h.s. comes from $\overline{\nabla}_x^\mu$. We can now express V_0^{CI} as $Z_{V^I}V_0^I$ and solve for Z_{V^I}. This estimate of the RC is free of leading $\mathcal{O}(a)$ errors.

We now turn to the WI of (75) which, expressed in terms of improved operators, can be used for the determination of the tree-level $\mathcal{O}(a)$ improved RC Z_{A^I} of the axial current. The subtlety is that the integration $\int d^4x$ of the WI requires inclusion of the contact terms arising from the δ-function of the rotated propagator of (84). Thus, for any value of the z-parameter, the Clover improved version of (75) becomes

$$\begin{aligned}&2\rho^{12}\int dx\int d\mathbf{y}\langle P^{12}(x)A_\nu^{31}(y)V_\rho^{23}(0)\rangle\\ &\quad=\left(\frac{Z_{V^I}^{23}}{Z_{A^I}^{31}Z_{A^I}^{12}}-\rho^{12}a\bar{z}\right)\int d\mathbf{y}\langle V_\nu^{32}(y)V_\rho^{23}(0)\rangle\\ &\quad-\left(\frac{Z_{A^I}^{13}}{Z_{A^I}^{12}Z_{V^I}^{23}}-\rho^{13}a\bar{z}\right)\int d\mathbf{y}\langle A_\nu^{31}(y)A_\rho^{13}(0)\rangle\end{aligned} \quad (88)$$

where $\bar{z}=(2z-1)$. All operators in the above equation, as well as the 2ρ factors, are to be understood as improved quantities. For 2ρ this is true provided that it is computed from (64) with improved operators A_μ^I and P^I and a symmetrized lattice temporal derivative $\overline{\nabla}^0$.

5.3 Comparison of the perturbative and WI estimates of the RC's

From the discussion so far, it should be clear that it is essential to the reliability of any WI computation, performed

Table 3. Estimates of the RC's of the vector current (Z_V), the axial current (Z_A) and the ratio of the scalar and pseudoscalar densities (Z_S/Z_P) from various perturbative and non-perturbative methods (see text for labels) at $\beta = 6.0$. All results are obtained at a small value of the light quark masses ($am \simeq 0.07$). The corresponding references are shown in square brackets

	Method	Wilson	Clover
Z_V	SPT	0.83	0.90
	MC-PB($\ln \Box$)	0.63	0.80
	Ratio (R_ρ)	0.57(1) [13]	0.82 [This work]
	Ratio (R_0)	0.73(3) [13]	0.82 [This work]
	WI (Axial)	0.79(4) [11]	0.80(2) [This work]
	NP	0.75(5) [24]	0.84(1) [22]
Z_A	SPT	0.87	0.98
	MC-PB($\ln \Box$)	0.72	0.95
	WI (Axial)	0.85(7) [11]	1.11(2) [This work]
	NP	0.80(5) [24]	1.06(8) [22]
Z_S/Z_P	SPT	1.10	1.20
	MC-PB($\ln \Box$)	1.34	1.54
	WI (Axial)	N/A	1.64(5) [This work]
	NP	N/A	1.6(1) [22]

at finite cutoff, that any scale with mass dimensions (such as Λ_{QCD}, masses m and momenta magnitudes q) must satisfy the condition

$$\Lambda_{QCD}, m, q \ll a^{-1} \tag{89}$$

In an ideal situation (g_0^2 and a extremely small) the above condition is satisfied and the estimates of the Z's obtained from different correlation functions with the WI/Ratio method should all agree. They should also agree with the Z estimates obtained in PT, carried out at sufficiently high order. The argument can now be turned upon its head: if the Z_O's obtained from WI's or ratios of different correlation functions are not in good agreement, this is to be interpreted as a signal that the calculation suffers from large discretization errors. In fact, the "dependence" of Z_O from the external states and/or the quark mass is a criterion used to estimate the finite lattice spacing effects which affect the lattice calculations. On the other hand, if the WI/Ratio result for Z_O turns out to be reliable, any discrepancies with its PT estimate is to be attributed to the truncation of the perturbative series.

To illustrate the point, we collect some results in Table 3, obtained at a typical value of the lattice coupling constant, $\beta = 6.0$. For the Wilson case, the presence of large $\mathcal{O}(a)$ effects is highlighted by the $20\% - 30\%$ variation in the various non-perturbative estimates of Z_V. This wide spread of values renders problematic their comparison to the perturbative estimates (both SPT and BPT). For Z_A, the non-perturbative values are compatible within statistical errors. However, these errors are too large to allow any conclusion on the accuracy of the two perturbative determinations.

For Clover fermions, the spread of the different WI estimates is reduced to less than 5%. This is due to the elimination of $\mathcal{O}(a)$ terms guaranteed by the Clover improvement. This result implies that lattice QCD calculations, performed with the Clover action at light quark mass, are significantly less affected by the finiteness of the lattice spacing than their Wilson counterparts. Note that BPT predictions for Z_V agree well with the non-perturbative results. However, for the ratio Z_S/Z_P BPT is less successful. Moreover, the PT estimate of Z_A is less than 1, whereas the NP value exceeds unity. BPT cannot rectify this discrepancy. Thus, $O(g^4)$ terms are probably important in these cases.

So far we have presented results at $\beta = 6.0$. The reason for this choice is that this coupling is typically used by present-day lattice QCD groups for calibration purposes. Most studies of an explorative nature (e.g. RC's from WI's, or from the NP method, with or without Clover improvement) have first been carried out at $\beta = 6.0$. Thus a comparison of the RC's obtained from all different methods is possible at this value of the coupling. In Subsect. 7.1 we also present results for Z_V obtained from the ratio method with the Wilson action at $\beta = 6.4$. Although the cutoff a^{-1} is considerably increased, we will see that finite cutoff effects remain sizeable. Thus, our qualitative conclusions on systematic errors characterizing the Wilson action remain valid at $\beta = 6.4$. Finally, finite RC's with the Clover action (at light quark mass) have also been computed at $\beta = 6.2$ [20].

6 Lattice-continuum normalization of fields and currents

We now examine a proposal due to Lepage, Mackenzie and Kronfeld [36] -[38] for the removal of $\mathcal{O}(am)$ systematic errors which are relevant in lattice calculations with the Wilson action at large quark masses. After a critical review of their recipe, we extend it to the Clover case. For both the Wilson and Clover cases, we apply it to the WI computation of Z_V and Z_A and discuss its merits and shortcomings.

First we set up our notation. So far we have worked with the lattice action $S = S_f + S_g$ of (8) and (9), which depends on lattice fields and bare parameters; i.e. it has the form $S[\bar{\psi}, \psi, U_\mu, m_0, g_0^2, a]$. In numerical simulations, it is convenient to work with dimensionless fermion fields and the hopping parameter instead of the quark mass; i.e. the action used in Monte Carlo simulations

$$\begin{aligned} S^{LAT} = &- \sum_{x,\mu} K \big[\bar{\chi}(x)(1-\gamma_\mu) U_\mu(x) \chi(x+\mu) \\ &+ \bar{\chi}(x+\mu)(1+\gamma_\mu) U_\mu^\dagger(x) \chi(x) \big] \\ &+ \sum_x \bar{\chi}(x)\chi(x) + S_g \end{aligned} \tag{90}$$

has the form $S^{LAT}[\bar{\chi}, \chi, U_\mu, K, g_0^2]$, where K and χ are defined through

$$\chi = \sqrt{\frac{a^3}{2K}}\, \psi \qquad , \qquad \frac{1}{2K} - 4 = am_0 \tag{91}$$

Note that the lattice spacing a has been scaled away in S^{LAT}. All the equations appearing in the previous sections can now be trivially rewritten in terms of $\bar{\chi}$ and χ fields and masses in lattice units $m_0 a$ (or K); the dependence on a drops out and the equations express relations between dimensionless quantities. From now on, all lattice quantities (such as quark propagators, operators and their correlation functions) are to be understood as defined in terms of the $\bar{\chi}$ and χ fields; i.e. they are (dimensionless) quantities expressed in lattice units.

In order to connect lattice fields to continuum (bare) ones, we must expand the lattice action for $a \to 0$ at fixed g_0 and K. This action then reduces to the continuum QCD action, S^{QCD}, provided we relate the continuum field A_μ to its lattice counterpart U_μ in the following way:

$$U_\mu = \exp(iag_0 A_\mu) \tag{92}$$

The standard prescription for the transcription of any continuum quantity on the lattice follows from (91) and (92). For example, in the naive continuum limit, the dimensionless lattice quark propagator $\mathcal{S}(x-y) = \langle \chi(x)\bar{\chi}(x)\rangle$ becomes its QCD counterpart $\mathcal{S}^{QCD}(x-y) = \langle \psi(x)\bar{\psi}(x)\rangle$ when normalized as follows:

$$a^{-3}\ 2K\ \mathcal{S}(x-y) \to \mathcal{S}^{QCD}(x-y) \tag{93}$$

Clearly, such transcriptions are not unique. Modifications of the lattice quantity by terms which are $\mathcal{O}(a)$ give equally acceptable transcriptions. For example, both the lattice local vector current $V_\mu(x) = \bar{\chi}_1(x)\gamma_\mu\chi_2(x)$ and the corresponding point-split conserved current $V_\mu^C(x)$ are good transcriptions of the vector current obtained from S^{QCD}:

$$\begin{aligned} a^{-3}\sqrt{2K_1 2K_2}\, V_\mu(x) &\to V_\mu^{QCD}(x) \\ a^{-3}\sqrt{2K_1 2K_2}\, V_\mu^C(x) &\to V_\mu^{QCD}(x) \end{aligned} \tag{94}$$

Note that these relations are just transcriptions of bare QCD quantities in terms of quantities derived from S^{LAT}. They are not relationships between lattice and renormalized quantities, which must involve the presence of RC's; e.g. $\hat{V}_\mu = Z_V V_\mu$. In our notation, the QCD superscript and the "hat" distinguish the two cases.

6.1 KLM factors for the Wilson action

The proposal by Lepage, Mackenzie and Kronfeld (KLM) [36]–[38] is a refinement of the above matching. Their aim is to account for large $\mathcal{O}(am)$ corrections in the limit of small aq (q stands for the magnitude of any relevant spatial momentum). In practice their recipe consists in calculating the matching factor between lattice correlation functions and their continuum counterpart in the free theory, for finite lattice spacing and vanishing spatial momentum. The claim is that this free theory correction should account for most of the $\mathcal{O}(am)$ systematic error also in the interacting case. For example, in the free theory, the matching between the continuum and lattice quark propagators in $(t, \mathbf{p})$-space at $\mathbf{p} = \mathbf{0}$ is

$$\mathcal{S}^{QCD}(t, \mathbf{p}=\mathbf{0}) = a^{-3} 2K(1+am_0)\mathcal{S}(t, \mathbf{p}=\mathbf{0}) \tag{95}$$

This implies for the lattice fermion field the matching relation

$$\psi = F_\psi(am_0)^{1/2} \sqrt{\frac{2K}{a^3}}\ \chi \tag{96}$$

where the KLM wave function normalization factor is

$$F_\psi(am_0) = 1 + am_0 \tag{97}$$

This factor tends to 1 in the limit $am_0 \to 0$ and the standard matching of (91) is recovered. We stress that this is a free theory matching which is due to the granularity of the space-time. It is not a renormalization.

A similar procedure applies to all correlation functions. For example consider the correlation function of the lattice local current between external quark states at zero spatial momentum,

$$L_\mu(t_x, t_y) = \sum_{\mathbf{x},\mathbf{y}} \langle \chi_1(y) V_\mu(x) \bar{\chi}_2(0)\rangle \tag{98}$$

It can easily be seen that, in the free theory and at finite cutoff, it is related to the continuum correlation function by

$$L_\mu^{QCD}(t_x, t_y) = a^{-6} 2K_1 2K_2 F_\psi(am_{01}) F_\psi(am_{02}) L_\mu(t_x, t_y) \tag{99}$$

where am_{01} and am_{02} are the bare quark masses of the two distinct flavours of the quark fields. Thus, there is no extra KLM normalization, apart from the one related to the quark fields. The KLM factor for the local vector current turns out to be 1. This is also the case of the local axial current and of any other local bilinear operator. The KLM factor for the spatial component of the conserved current $V_k^C(x)$ is also 1. On the other hand, as pointed out in [54], the temporal component of the conserved current $V_0^C(x)$ requires an extra KLM factor

$$F_{V_0^C}(am_{01}, am_{02}) = \left(1 + \frac{am_{01} + am_{02}}{2}\right)^{-1} \tag{100}$$

It arises from the matching of

$$C_0(t_x, t_y) = \sum_{\mathbf{x},\mathbf{y}} \langle \chi_1(y) V_0^C(x) \bar{\chi}_2(0)\rangle \tag{101}$$

to $C_0^{QCD}(t_y, t_x)$, which gives:

$$\begin{aligned} C_0^{QCD}(t_y, t_x) = a^{-6} 2K_1 2K_2 F_\psi(am_{01}) F_\psi(am_{02}) \\ \times F_{V_0^C}(am_{01}, am_{02}) C_0(t_y, t_x) \end{aligned} \tag{102}$$

It is now trivial to derive the KLM matching factors of hadronic correlation functions. We just have to consider the number of quark fields and the composite operators entering in the correlation function. For example, for the 2-point function of the vector currents, with the ρ-meson quantum numbers, we have

$$\begin{aligned} \sum_{\mathbf{x}} \langle V_k^C(x) V_k^\dagger(0)\rangle^{QCD} = a^{-6} 2K_u 2K_d F_\psi(am_{0u}) F_\psi(am_{0d}) \\ \times \sum_{\mathbf{x}} \langle V_k^C(x) V_k^\dagger(0)\rangle \end{aligned} \tag{103}$$

whereas for the vector 3-point function of the $D \to K$ decay we have

$$\sum_{\mathbf{x},\mathbf{y}} \langle P_K(0) V_0^C(x) P_D^\dagger(y)\rangle^{QCD} \qquad (104)$$
$$= a^{-6} 2K_u 2K_s 2K_c F_\psi(am_{0u}) F_\psi(am_{0s}) F_\psi(am_{0c})$$
$$\times F_{V_0^C}(am_{0s}, am_{0c}) \sum_{\mathbf{x},\mathbf{y}} \langle P_K(0) V_0^C(x) P_D^\dagger(y)\rangle$$

The KLM factors obtained above depend on the bare quark mass of (91). A possible improvement, which could take into account interaction effects, consists in using a more physical definition of the quark mass in the KLM factors. A natural choice is, for instance the subtracted quark mass m which, in terms of the hopping parameter K, is given by

$$am = \frac{1}{2K} - \frac{1}{2K_C} \qquad (105)$$

with the critical value K_C determined non-perturbatively. Unless otherwise stated, all KLM factors used in our numerical analysis are calculated at the above value of the quark mass (e.g. $F_\psi(am)$, $F_{V_0^C}(am)$ etc.). Another proposal has been made in [29], based on the observation that in lattice perturbation theory, a large fraction of renormalization effects comes from tadpole diagrams. The authors of [29] propose to take the bulk of these effects into account by implementing a Mean Field Tadpole Improved (MFTI) prescription. MFT Improvement amounts to the following two substitutions:
(1) The link variable in any operator, such as V_μ^C, is to be substituted by

$$U_\mu \to \frac{U_\mu}{u_0} \qquad (106)$$

where u_0 is any reasonable mean-field estimate of the expectation value of the link. Two standard choices are $u_0 = [\frac{1}{3} Tr U_P]^{1/4}$ and $u_0 = 1/(8K_c)$. In practice, for current values of the coupling constant, the two u_0 estimates differ by about 10% (about 2% for the Clover action). In the results of this work, the latter estimate of u_0 has been used.
(2) The hopping parameter is obtained by substituting

$$K \to \tilde{K} = K u_0 \qquad (107)$$

This implies, for example, that the MFTI quark mass is obtained by substituting

$$am_0 \to a\tilde{m} = \frac{1}{2\tilde{K}} - 4 = 8K_C(\frac{1}{2K} - \frac{1}{2K_C})$$
$$= 8K_C am \qquad (108)$$

The "standard" MFTI normalization of the quark field is given by the factor $\sqrt{2\tilde{K}}$; its KLM/MFTI correction is obtained through the substitution

$$\sqrt{2K} F_\psi(am) \to \sqrt{2\tilde{K}} \tilde{F}_\psi(a\tilde{m}) = \sqrt{2\tilde{K}(1 + a\tilde{m})} \qquad (109)$$

while for the conserved current V_0^C, the KLM/MFTI factor becomes

$$F_{V_0^C} \to \tilde{F}_{V_0^C} = \frac{1}{u_0} \left(1 + \frac{a\tilde{m}_1 + a\tilde{m}_2}{2}\right)^{-1} \qquad (110)$$

It must be kept in mind that the $1/u_0$ factor comes from the link U_μ present in the conserved current (i.e. it is a redefinition of the lattice operator due to the redefinition of the gauge field through (106)) whereas the mass dependent factor arises from (108) (i.e. it is a rescaling of the bare mass). This difference in the origin of the two factors will be important when we compute ME's of the conserved currents with KLM/MFTI corrections (see comments after (124)). We also note that the prescription of (108) has also been obtained by a tadpole resummation argument, without recourse to MFTI, in [42].

The KLM factors, having been calculated in the free theory and at zero quark spatial momentum, do not account for, say, $\mathcal{O}(g_0^2 am)$, $\mathcal{O}(a\Lambda_{QCD})$ and $\mathcal{O}(ap)$ corrections. The hope is that at large quark mass and small coupling, they account for most of the systematic error due to the finiteness of the UV cutoff. In Sect. 7 we will test these expectations.

6.2 KLM factors for the Clover action

We now calculate the KLM factors for the Clover case. Since the free fermion propagator is the same for both the Wilson and Clover actions, its KLM factor F_ψ is the same. However, fermion fields need to be rotated, as for example in (79). Since we usually opt for this specific rotation ($z = 1$), most correlation functions are expressed in terms of the effective lattice propagator $\mathcal{S}^{eff}(x-y)$, given by (83). The KLM factor for the effective propagator is then given by

$$\mathcal{S}^{QCD}(t, \mathbf{p}=\mathbf{0}) = a^{-3} \frac{2K(1+am_0)}{\left[1 + \frac{1}{4}\left(1 + am_0 - \frac{1}{1+am_0}\right)\right]^2}$$
$$\times \mathcal{S}^{eff}(t, \mathbf{p}=\mathbf{0}) \qquad (111)$$

where again everything is worked out in the free case. Thus, compared to the Wilson case of (95), there is an extra term, $F_R(am_0)$, arising from the fermion rotations in the improved KLM factor:

$$F_\psi^I(am_0) = F_\psi(am_0) F_R(am_0)$$
$$= (1 + am_0) F_R(am_0) \qquad (112)$$

with

$$F_R(am_0) = \left[1 + \frac{1}{4}\left(1 + am_0 - \frac{1}{1+am_0}\right)\right]^{-2} \qquad (113)$$

It can be seen immediately that the overall KLM factor, $F_\psi^I(am_0)$, is $1 + \mathcal{O}(a^2 m_0^2)$, in accordance with $\mathcal{O}(a)$ improvement. From now on, we will directly substitute the

bare mass m_0 by the subtracted one m, in the spirit of (105).

Let us consider the quark correlation function of the local improved current $V_\mu^I = \bar{\chi}_1^R \gamma_\mu \chi_2^R$ (i.e. the improved analogue of (98)):

$$L_\mu^I(t_x, t_y) = \sum_{\mathbf{x},\mathbf{y}} \langle \chi_1^R(y) V_\mu^I(x) \bar{\chi}_2^R(0) \rangle \tag{114}$$

It is matched to its continuum counterpart through

$$L_\mu^{QCD} = a^{-6} F_\psi^I(am_1) F_\psi^I(am_2) L_\mu^I(t_x, t_y) \tag{115}$$

Thus we see that also in the Clover case the KLM factor for the local vector current, as well as for any other local bilinear operator, is 1.

We now pass to the KLM factors of the conserved improved current of (86). Unlike the improved local current V_μ^I, fermion fields are not rotated for this current. We first consider the improved quark correlation function of the current's temporal component V_0^{CI}:

$$C_0^I(t_y, t_x) = \sum_{\mathbf{x},\mathbf{y}} \langle \chi_1^R(y) V_0^{CI}(x) \bar{\chi}_2^R(0) \rangle \tag{116}$$

The matching factors for this correlation are easily found to be:

$$C_0^{QCD}(t_y, t_x) = a^{-6} 2K_1 2K_2 F_\psi(am_1) F_\psi(am_2) F_R^{1/2}(am_1) \times F_R^{1/2}(am_2) F_{V_0^C}(am_1, am_2) C_0^I(t_y, t_x) \tag{117}$$

where $F_{V_0^C}$ is given in (100). Note that at $\mathbf{p} = \mathbf{0}$ the $\nabla_\mu \bar{\chi} \sigma_{\mu 0} \chi$ term of the current vanishes. The above overall KLM factor in (117) is $1 + \mathcal{O}(a^2 m^2)$, in accordance with improvement.

Finally, we examine the improved correlation $C_k^I(t_x, t_y)$ of the spatial component of the conserved improved current:

$$C_k^I(t_y, t_x) = \sum_{\mathbf{x},\mathbf{y}} \langle \chi_1^R(y) V_k^{CI}(x) \bar{\chi}_2^R(0) \rangle \tag{118}$$

The standard KLM calculation yields the following matching:

$$C_k^{QCD}(t_y, t_x) = a^{-6} 2K_1 2K_2 F_\psi(am_1) \times F_\psi(am_2) F_R^{1/2}(am_1) \times F_R^{1/2}(am_2) F_{V_k^{CI}}(am_1, am_2) C_k^I(t_y, t_x) \tag{119}$$

where

$$F_{V_k^{CI}} = \left[1 - \frac{1 - (1 + am_1)^2 (1 + am_2)^2}{4(1 + am_1)(1 + am_2)} \right]^{-1} \tag{120}$$

Again, being $1 + \mathcal{O}(a^2 m^2)$, the overall KLM factor in (119) is consistent with improvement.

We conclude by noting that using the Clover version of the axial WI, in the spirit of KLM, requires a matching factor also for 2ρ. From the improved version of (64), with a symmetric lattice temporal derivative $\overline{\nabla}^0$, we obtain the KLM matching

$$2\rho^{QCD} = \frac{2(am_1 + am_2)}{\exp(am_1 + am_2) - \exp(-am_1 - am_2)} 2\rho^I \tag{121}$$

Just like in the Wilson case, $\tilde{m}$'s and u_0's can be introduced, whenever required by the prescription, in order to account for KLM/MFTI matching factors. Again, the extension of the Clover KLM factors to those characterizing hadronic correlation functions is straightforward.

7 Non-perturbative results for the RC's

In this section we present results for the RC's Z_V and Z_A and the ratio Z_S/Z_P obtained, non-perturbatively, from numerical studies of the chiral WI's on the lattice. The RC of the local vector current, Z_V, has been also computed by taking the ratio of ME's of the conserved to the local vector current between different external states.

The main purpose of this study is to investigate the systematic errors present in these calculations and, in particular, those due to the finiteness of the lattice cutoff. For this purpose, we have considered both the standard Wilson action and the tree-level $\mathcal{O}(a)$-improved Clover action. In order to enhance the finite cutoff effects, we have studied a large range of values of the lattice bare quark masses, which vary from $am \simeq 0.05$ up to $am \simeq 1$. As expected, we find that these effects are typically larger with the Wilson action than with the Clover action.

Moreover, we have also tried to correct the finite cutoff effects by following the KLM prescriptions, discussed in the previous section. We find that KLM corrections are typically more effective in the Wilson case, where they are $\mathcal{O}(am)$ rather than in the Clover case, where they are $\mathcal{O}(a^2 m^2)$. The analysis of the Clover RC's shows that the bulk of the finite cutoff effects depends linearly on the bare quark mass. This suggests the existence of large $\mathcal{O}(g^2 am)$ terms, which, being a consequence of the quantum loops, cannot be corrected by the tree-level KLM prescriptions.

A significant result is that the finite cutoff effects, even in the Clover case, can be as large as $20 - 30\%$, for values of the bare quark masses which are currently used in numerical simulations (e.g. $am \simeq 0.5$). This result strongly supports the necessity of an additional systematic improving strategy, such as [32,34,35]. In the following we present our results in detail.

7.1 Ratio determination of Z_V with the Wilson action

We first present a computation of the RC of the local vector current, Z_V, in the case of the standard Wilson action. The data have been obtained by averaging over 20 quenched gauge field configurations, generated at $\beta = 6.4$. This value of the lattice coupling constant corresponds to a relatively large value of the lattice cutoff, $a^{-1} \simeq 3.6 - 3.8$

GeV. The lattice volume is $L^3 \times T = 24^3 \times 60$. The same set of gauge configurations and quark propagators has already been used in previous phenomenological studies of heavy meson physics on the lattice [55].

Z_V has been computed by considering different ratios of two- and three-point correlation functions which reduce, asymptotically in time, to different ME's of the conserved and the local vector currents; cf. (73). For simplicity, these ME's will be denoted in the following as $< 0|V_k|\rho >$, $< K|V_\mu|D >$ and $< K^*|V_\mu|D >$, regardless of the specific flavour content of the mesons in the external states. In particular, by D we mean a pseudoscalar meson consisting of a "light" and a "heavy" quark, whereas by K, K^* and ρ we indicate mesons composed by two light quarks. In all cases, the Wilson parameter of the light quark mass is fixed at $K = 0.1485$, the critical value being $K_C = 0.1506$. Thus this mass corresponds to a bare quark mass $am \simeq 0.05$. The heavy quark mass varies in the range between $am \simeq 0.2$ and $am \simeq 0.6$. In the three point correlation functions, the source of the final K and K^* mesons is inserted at the time $t = 0$. The D meson is at $t = 28$ and the vector current varies in the interval $12 \leq t \leq 16$. The D meson is always considered in its rest frame and the spatial momentum carried by the current is denoted by $\mathbf{q}$. For simplicity, the momentum transfers $\mathbf{q} = (0, 0, 0)$ and $\mathbf{q} = (1, 0, 0)$ (expressed in lattice units of $2\pi/La$) will be abbreviated as $q = 0$ and $q = 1$ respectively.

A first way of obtaining Z_V consists in computing the ratio R_0 of three point correlation functions defined in (73). This ratio has been computed at momentum transfer $q = 0$. In the large time limit, it is given by:

$$R_0(q=0) = \frac{< K|V_0^C|D >}{< K|V_0|D >} = Z_V + \mathcal{O}(a) \qquad (122)$$

According to the KLM claim, the bulk of the $\mathcal{O}(a)$ corrections in this expression should be given by the KLM factor $F_{V_0^C}(am_1, am_2)$ of (100) associated with the vertex of the conserved current V_0^C (the KLM factors $F_\psi(am)$, associated with the external quark fields, cancel in R_0). Therefore we also define the ratio:

$$\begin{aligned} R_0^{KLM} &= R_0 \cdot F_{V_0^C}(am_1, am_2) \\ &= R_0 \cdot \left(1 + \frac{am_1 + am_2}{2}\right)^{-1} \\ &= Z_V + \mathcal{O}(a) \end{aligned} \qquad (123)$$

If the KLM prescription works in this case, the $\mathcal{O}(a)$ terms in (123) should be smaller than those in (122). This means that the ratio R_0^{KLM} should exhibit a milder dependence on the two quark masses. In addition, we also consider the MFTI version of the ratio R_0. It is obtained from (123), by substituting the bare quark masses with their MFTI expressions:

$$R_0^{MFTI} = R_0 \cdot \left(1 + \frac{a\tilde{m}_1 + a\tilde{m}_2}{2}\right)^{-1} = Z_V + \mathcal{O}(a) \quad (124)$$

Notice that, with respect to the definition of the KLM/MFTI factor $\tilde{F}_{V_0^C}$ given in (110), we have omitted in the previous equation a factor u_0. As pointed out after (110), its presence would imply a redefinition of the conserved current, whereas we are only interested in a comparison of the discretization errors with and without KLM/MFTI corrections. Therefore, by omitting u_0, we are computing the ratio R_0^{MFTI} of the ME's of the same operators which appear in the two ratios R_0 and R_0^{KLM}. In this way, all three ratios give estimates of the same Z_V and only differ by $\mathcal{O}(a)$ effects.

The values of the three ratios R_0 are shown in Table 4, for different values of the hopping parameter. They are also plotted in Fig. 1 as a function of the "effective" quark mass, $am_{eff} = (am_1 + am_2)/2$. We see that R_0 has a strong dependence on the bare quark masses, varying by approximately 50% as the mass varies from $am \simeq 0.2$ to $am \simeq 0.6$. This dependence is however significantly reduced when we apply the KLM correction. This reduction is somewhat bigger in the MFTI prescription.

Table 4. The ratios R_0, R_0^{KLM} and R_0^{MFTI} for the Wilson action and for different values of the quark masses. In the three point correlation functions the light spectator quark has mass $am \simeq 0.05$. The statistical errors in the table are less than (or equal to) 1 on the last digit

K_1	K_2	am_{eff}	R_0	R_0^{KLM}	R_0^{MFTI}
0.1275	0.1275	0.602	1.278	0.798	0.741
0.1275	0.1325	0.528	1.207	0.790	0.738
0.1275	0.1375	0.459	1.141	0.782	0.735
0.1325	0.1325	0.454	1.136	0.782	0.735
0.1275	0.1425	0.395	1.081	0.775	0.732
0.1325	0.1375	0.385	1.070	0.773	0.731
0.1275	0.1485	0.324	1.017	0.768	0.731
0.1325	0.1425	0.321	1.010	0.764	0.728
0.1375	0.1375	0.316	1.005	0.763	0.728
0.1375	0.1425	0.253	0.945	0.754	0.725
0.1325	0.1485	0.250	0.946	0.756	0.727
0.1425	0.1425	0.189	0.885	0.744	0.721
0.1375	0.1485	0.182	0.880	0.745	0.722
0.1425	0.1485	0.118	0.820	0.733	0.718

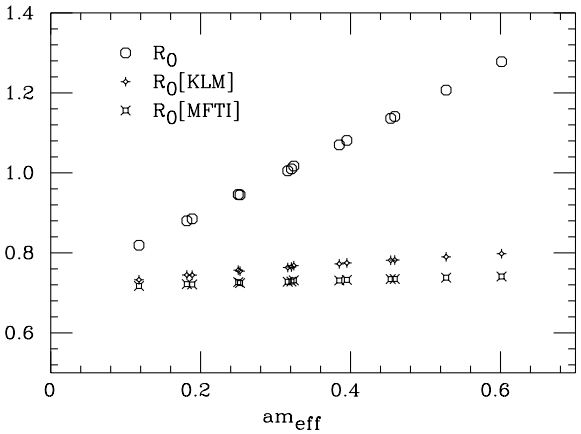

Fig. 1. The various ratios R_0, defined in (122), (123) and (124) for Wilson fermions, as functions of the effective quark mass

In order to discuss these results in a more quantitative way, we fit the three ratios R_0, as a function of the quark masses, taking into account contributions up to quadratic terms:

$$R_0 = A+B(am_1+am_2)+C(am_1+am_2)^2+D(am_1-am_2)^2 \tag{125}$$

Since the current vertex is symmetric under interchange of the two quark fields, there is no linear dependence on the mass difference $(am_1 - am_2)$. For all three ratios R_0, R_0^{KLM} and R_0^{MFTI} the value of the coefficient A in (125) gives an estimate of Z_V, with presumably negligible $\mathcal{O}(am)$ systematic errors, which we denote by Z_V^{eff}. The result in all three cases is approximately the same and we quote, as an overall estimate:

$$Z_V^{eff} = 0.71(1) \quad , \quad \text{from} \;\; R_0 \tag{126}$$

The parameter B in (125) gives the size of the leading $\mathcal{O}(am)$ corrections. We find that B decreases from $B \simeq 0.46$ for R_0, to $B \simeq 0.09$ for R_0^{KLM} and to an even smaller value, $B \simeq 0.03$, for R_0^{MFTI}. The coefficients C and D of the quadratic terms are always of the order 10^{-2} or smaller in magnitude. Thus, in the case of the ratio R_0, the KLM prescription correctly takes into account and removes the bulk of the finite cutoff effects.

Next we compare results for Z_V obtained from different ratios of correlation functions, cf. (73). In particular, for different values of the quark masses, we have computed:

$$\begin{aligned} R_\rho(q=0) &= \frac{\sum_r \epsilon_r^k \langle 0|V_k^C(0)|\rho_r\rangle}{\sum_r \epsilon_r^k \langle 0|V_k(0)|\rho_r\rangle} = Z_V + \mathcal{O}(a) \\ R_k(q=1) &= \frac{\langle K|V_k^C(0)|D\rangle}{\langle K|V_k(0)|D\rangle} = Z_V + \mathcal{O}(a) \\ R_k^*(q=1) &= \frac{\sum_r \epsilon_r^\lambda \langle K_r^*|V_k^C(0)|D\rangle}{\sum_r \epsilon_r^\lambda \langle K_r^*|V_k(0)|D\rangle} = Z_V + \mathcal{O}(a) \end{aligned} \tag{127}$$

The KLM corrections to the correlation functions of these ratios always cancel out between numerator and denominator. Thus, according to the KLM expectations, all these ratios should be free of large $\mathcal{O}(am)$ corrections.

The values of R_ρ, R_k and R_k^* are given in Table 5 for different values of the vector current quark masses. The same ratios are also plotted in Fig. 2 as a function of the effective quark mass. In the figure, the ratio R_0^{KLM} is also plotted for comparison. We see that R_ρ, R_k and R_k^* exhibit quite a different behaviour as a function of the quark masses. When the mass varies from $am \simeq 0.2$ to $am \simeq 0.6$, the ratios R_ρ and R_k^* remain almost constant, while the ratio R_k is found to be affected by large $\mathcal{O}(a)$ corrections. In particular, by performing a fit as in (125), we find a large contribution coming from the term proportional to $(am_1-am_2)^2$, with a coefficient $D \simeq -0.51$. Compared to its values corresponding to the three previous estimates of R_0, D of R_k is larger by approximately an order of magnitude. The total finite cutoff effects in the ratio R_k can be as large as 50% for $am \simeq 0.6$. These effects cannot be corrected by the KLM prescription.

Table 5. The ratios R_ρ, R_k and R_k^* for the Wilson action at different values of the vector current quark masses. In the three point correlation functions the light spectator quark has mass $am \simeq 0.05$

K_1	K_2	am_{eff}	R_ρ	R_k	R_k^*
0.1275	0.1275	0.602	0.622(1)	1.073(41)	0.643(23)
0.1275	0.1325	0.528	0.628(1)	0.972(37)	0.639(23)
0.1275	0.1375	0.459	0.633(1)	0.877(34)	0.634(23)
0.1325	0.1325	0.454	0.632(1)	0.979(38)	0.634(23)
0.1275	0.1425	0.395	0.638(1)	0.792(30)	0.628(22)
0.1325	0.1375	0.385	0.637(1)	0.883(34)	0.630(23)
0.1275	0.1485	0.324	0.643(2)	0.719(20)	0.625(21)
0.1325	0.1425	0.321	0.642(1)	0.795(30)	0.624(23)
0.1375	0.1375	0.316	0.641(1)	0.891(34)	0.625(23)
0.1375	0.1425	0.253	0.645(1)	0.801(30)	0.620(23)
0.1325	0.1485	0.250	0.646(2)	0.721(20)	0.623(22)
0.1425	0.1425	0.189	0.649(1)	0.812(30)	0.616(23)
0.1375	0.1485	0.182	0.649(2)	0.723(19)	0.621(22)
0.1425	0.1485	0.118	0.653(2)	0.727(19)	0.618(23)
0.1485	0.1485	0.047	0.658(2)	—	—

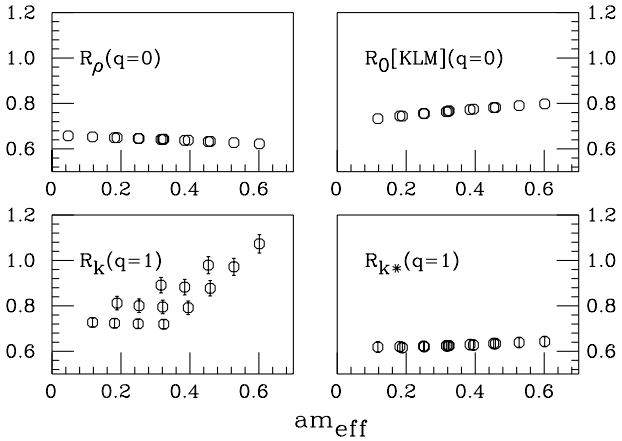

Fig. 2. The ratios R_ρ, R_0^{KLM}, R_k and R_k^*, for Wilson fermions, as functions of the effective quark mass

The values of the coefficients A of the fits give an estimate of Z_V with presumably negligible $\mathcal{O}(am)$ effects. We quote:

$$\begin{aligned} Z_V^{eff} &= 0.66(1) \quad , \quad \text{from} \;\; R_\rho \\ Z_V^{eff} &= 0.68(1) \quad , \quad \text{from} \;\; R_k \\ Z_V^{eff} &= 0.62(1) \quad , \quad \text{from} \;\; R_k^* \end{aligned} \tag{128}$$

The differences between these values and the R_0 estimate of (126) for Z_V represent a measure of the $\mathcal{O}(a)$ corrections which remain after the $\mathcal{O}(am)$ terms have been presumably removed by the fit. These corrections, which are expected to be $\mathcal{O}(aq)$ or $\mathcal{O}(a\Lambda_{QCD})$, leave a systematic uncertainty which is approximately 15%.

Table 6. The RC's Z_V, Z_A and the ratio Z_S/Z_P for the Clover action with light quark masses. The labels "g.i." and "g.d" distinguish results obtained from gauge-invariant and gauge-dependent WI's respectively

K_l	am_l	Z_V	Z_A (g.i.)	Z_A (g.d.)	Z_P/Z_S
0.1410	0.111	0.833(1)	1.06(5)	1.07(7)	0.61(3)
0.1425	0.073	0.824(2)	1.09(3)	1.14(8)	0.60(2)
0.1432	0.056	0.819(2)	1.07(7)	1.17(13)	0.60(3)

7.2 Ratio determination of Z_V, Z_A and Z_P/Z_S with the Clover action and light quark masses

In this subsection we present the results of a calculation of the RC's Z_{V^I}, Z_{A^I} and the ratio Z_{P^I}/Z_{S^I} obtained with the tree-level $\mathcal{O}(a)$-improved Clover action, by using the lattice chiral WI's. From now on, the superscript "I" of the Z_{O^I}'s will be dropped. We limit ourselves to light values of the bare quark masses. The value of the lattice coupling constant is $\beta = 6.0$. The lattice volume is $16^3 \times 32$. We have generated 18 quenched gauge field configurations. We have worked at three values of the Wilson hopping parameters, namely $K = 0.1410$, 0.1425 and 0.1432 corresponding to bare quark masses ranging from $am = 0.056$ to $am = 0.111$. The critical value is $K_C = 0.14554$.

All two- and three-point correlation functions have been computed by considering degenerate quark masses. The results are a replication of those of [17] for three quark masses. The RC of the local axial current Z_A and the ratio Z_P/Z_S have been obtained from gauge invariant chiral WI's, between external hadron states (see (69) and (70)). Gauge dependent chiral WI's between external quark states have also been implemented for a calculation of Z_A (see (57)). The RC of the local vector current, Z_V, has been computed from the ratio R_ρ, with zero spatial momenta. The results of this computation are presented in Table 6.

We see that, in the case of Z_V, the statistical errors are sufficiently small to reveal the presence of finite cutoff effects in the calculation. By performing a fit as in (125) we can obtain the value of Z_V, with presumably negligible am effects. We quote:

$$Z_V^{eff} = 0.80(1) \qquad , \qquad \text{from} \quad R_\rho \tag{129}$$

As seen in Table 6, the variations of the RC Z_A and the ratio Z_S/Z_P with the light quark masses are of the size of the statistical errors. Thus, in these cases the presence of finite cutoff effects cannot be determined. An analogous study at $\beta = 6.2$ has been performed by the UKQCD collaboration [20].

7.3 Ratio determination of Z_V with the Clover action and heavy quark masses

In order to investigate the size of finite cutoff effects when the tree-level $\mathcal{O}(a)$-improved Clover action is used, we have also computed Z_V at several large values of the quark mass. We have generated 30 quenched gauge field configurations at $\beta = 6.0$. The lattice volume is $16^3 \times 48$. Two different values of the light quark hopping parameter have been considered ($K = 0.1425$ and $K = 0.1432$) and four different values of the heavy quark mass. The latter correspond to bare quark masses varying from $am \simeq 0.3$ up to $am \simeq 0.9$.

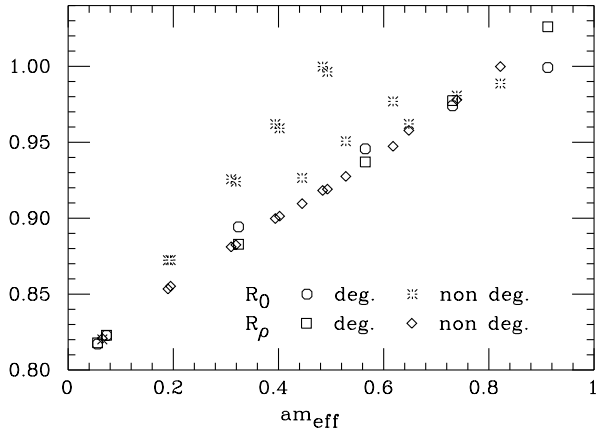

Fig. 3. The ratios R_0 and R_ρ, for Clover fermions, as a function of the effective quark mass. We distinguish between the cases of degenerate and non-degenerate quarks in the current

We have computed Z_V from the ratios R_0 and R_ρ of (73), with quark propagators improved according to (83). All external states and currents are always at zero spatial momentum. In the three-point correlation functions the pseudoscalar D meson is inserted in the middle of the lattice, at $t = 24$, so that results from both positive and negative times can be averaged, thus improving the statistics.

In order to investigate the effects of the KLM corrections in the Clover case, we have also considered the corresponding KLM-improved versions of the ratios R_0 and R_ρ. According to (115), (117) and (119), these corrections must be defined as:

$$\begin{aligned} R_0^{KLM} &= \frac{F_R(am_1)^{1/2} F_R(am_2)^{1/2}}{F_{V_0^C}(am_1, am_2)} \cdot R_0 \\ &= Z_V + \mathcal{O}(g^2 a, a^2) \\ R_\rho^{KLM} &= \frac{F_R(am_1)^{1/2} F_R(am_2)^{1/2}}{F_{V_k^{CI}}(am_1, am_2)} \cdot R_\rho \\ &= Z_V + \mathcal{O}(g^2 a, a^2) \end{aligned} \tag{130}$$

where the several KLM factors are given in (100), (113) and (120). Notice that, in the Clover case, the KLM corrections are always of $\mathcal{O}(a^2)$.

The values of R_0, R_ρ and their KLM counterparts are given in Table 7 for several values of the quark masses. The same ratios are also plotted in Figs. 3 and 4, with and without the KLM corrections, as a function of the effective mass $am_{eff} = (am_1 + am_2)/2$. The main feature shown in Fig. 3 is that both R_0 and R_ρ receive a large contribution from a term which is linear in the quark mass. This term affects the value of the two ratios by approximately 15%

Table 7. The ratios R_0 and R_ρ, and their KLM-improved counterparts, for the Clover action at different values of the quark masses. The statistical errors are always less than (or equal to) 1 on the last digit

K_1	K_2	am_{eff}	R_0	R_0^{KLM}	R_ρ	R_ρ^{KLM}
0.1150	0.1150	0.912	0.999	0.949	1.026	1.009
0.1150	0.1200	0.822	0.989	0.942	1.000	0.991
0.1150	0.1250	0.739	0.981	0.936	0.978	0.976
0.1200	0.1200	0.731	0.974	0.934	0.977	0.974
0.1200	0.1250	0.648	0.962	0.926	0.958	0.959
0.1150	0.1330	0.618	0.977	0.929	0.947	0.950
0.1250	0.1250	0.565	0.946	0.917	0.937	0.941
0.1200	0.1330	0.528	0.951	0.916	0.927	0.932
0.1150	0.1425	0.493	0.996	0.931	0.919	0.922
0.1150	0.1432	0.484	0.999	0.932	0.918	0.920
0.1250	0.1330	0.445	0.927	0.902	0.910	0.916
0.1200	0.1425	0.402	0.959	0.913	0.901	0.903
0.1200	0.1432	0.394	0.962	0.914	0.899	0.902
0.1330	0.1330	0.324	0.894	0.881	0.883	0.889
0.1250	0.1425	0.319	0.924	0.893	0.882	0.885
0.1250	0.1432	0.310	0.926	0.894	0.881	0.883
0.1330	0.1425	0.195	0.872	0.861	0.855	0.858
0.1330	0.1432	0.190	0.872	0.860	0.853	0.855
0.1425	0.1425	0.073	0.823	0.822	0.823	0.824
0.1425	0.1432	0.065	0.820	0.819	0.821	0.821
0.1432	0.1432	0.056	0.817	0.817	0.818	0.818

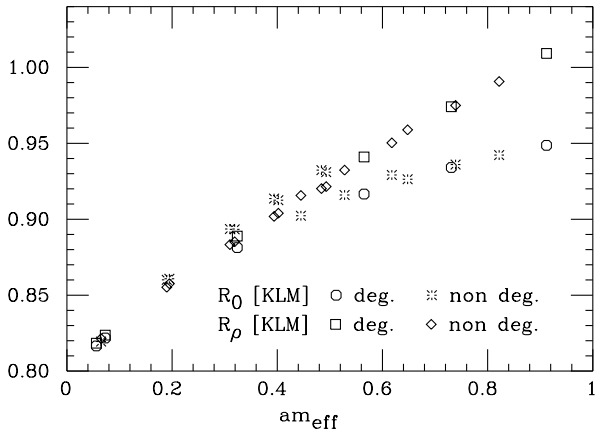

Fig. 4. Same as in Fig. 3 with KLM corrections

at $am \simeq 0.5$ and by about 25% at $am \simeq 0.9$. Moreover, such a large systematic error cannot be corrected by the KLM factors, the latter being at least quadratic in the quark masses. In fact, a qualitatively similar behaviour is found for the points in Fig. 4, where the KLM correction has been implemented.

By looking at Fig. 3 we also notice that the values of R_0, obtained with degenerate quark masses of the vector current, and those obtained by non-degenerate quarks, do not lie on the same curve. This suggests a significant dependence of this ratio on terms quadratic in the mass difference $(am_1 - am_2)^2$. In this respect, Fig. 4 shows that the KLM correction improves the situation, since the difference between degenerate and non-degenerate points is largely reduced. However, the KLM prescription cannot correct large $\mathcal{O}(g^2 am)$ effects, which are evidently present in Fig. 4. It would also appear from this figure that such terms are independent of the ME's, since the difference between R_0 and R_ρ settles only after masses of approximately $am \simeq 0.5$.

Table 8. Values of the coefficients A, B, C and D obtained from a fit of the ratios R_0, R_ρ and their KLM-improved counterparts

	A	B	C	D
R_0	0.80	0.19	-0.05	0.09
R_0^{KLM}	0.80	0.16	-0.05	0.03
R_ρ	0.81	0.12	-0.0008	-0.002
R_ρ^{KLM}	0.80	0.14	-0.02	-0.008

To make this discussion more quantitative, we fit the ratios R_0, R_ρ and their KLM counterparts, as in (125). The resulting values of the coefficients A, B, C and D are given in Table 8. The values of the coefficient A, in Table 8, represent an estimate of Z_V which is presumably free from any $\mathcal{O}(am)$ corrections. We then quote:

$$Z_V^{eff} = 0.80(1) \qquad , \qquad \text{from } R_0 \ , \ R_\rho \tag{131}$$

in remarkable agreement with its determination from light quark masses; cf. (129). Notice that the two results have been obtained from completely independent numerical simulations.

The values of the coefficients B in Table 8 provide an estimate of the size of the correction coming from a term which is linear in the quark mass. Since the action is tree-level improved, this term must be of $\mathcal{O}(g^2 a)$ or higher. We find $B \sim 0.15$, for both ratios R_0 and R_ρ. As expected, these coefficients are practically unaffected by the KLM corrections. Remarkably, however, we find that their value is almost "universal", in the sense that it is almost independent on the external states of the ME of the vector current.

On the other hand, the values of the coefficients C and D of the quadratic terms are not universal. For the ratio R_0, the large value of coefficient D is found to be significantly reduced by the KLM correction.

We stress again that all these $\mathcal{O}(a)$ effects are quite relevant in the region which, for current values of the lattice spacing, corresponds approximately to the charm quark mass. This in turn is crucial for the phenomenological studies of the heavy flavour physics on the lattice.

In an attempt to reduce the large $\mathcal{O}(g^2 a)$ effects observed in the data, we have also tried to modify the definition of the improved operators by varying the z parameter in (84). Specifically, we have considered the values $z = 0$ and $z = 0.5$ besides the standard choice $z = 1$ analyzed so far. We find that in all cases the size of the $\mathcal{O}(g^2 a)$ contributions is approximately the same. Thus, no specific choice of z appears to improve the situation.

Table 9. The RC Z_V, for the Clover action, obtained from the axial WI with and without the KLM correction. For comparison, the values of the ratio R_0^{KLM} are also shown in the last column

K_1	K_2	am_{eff}	Z_V	Z_V^{KLM}	Z_V (R_0^{KLM})
0.1150	0.1150	0.912	0.62(1)	1.01(1)	0.949
0.1200	0.1200	0.731	0.68(1)	0.93(1)	0.934
0.1250	0.1250	0.565	0.74(1)	0.88(1)	0.917
0.1330	0.1330	0.324	0.79(1)	0.84(1)	0.881
0.1425	0.1425	0.073	0.80(2)	0.80(2)	0.822
0.1432	0.1432	0.056	0.79(2)	0.79(2)	0.817

Table 10. The RC Z_A for the Clover action, with and without KLM corrections, for different values of the quark masses

K_1	K_2	am_{eff}	Z_A	Z_A^{KLM}
0.1150	0.1150	0.912	0.87(1)	1.07(1)
0.1200	0.1200	0.731	0.92(1)	1.05(1)
0.1250	0.1250	0.565	0.97(1)	1.05(1)
0.1150	0.1432	0.484	0.88(1)	0.93(1)
0.1200	0.1432	0.394	0.91(1)	0.95(1)
0.1330	0.1330	0.324	1.04(1)	1.06(1)
0.1250	0.1432	0.310	0.95(1)	0.97(1)
0.1330	0.1432	0.190	1.01(1)	1.02(1)
0.1425	0.1425	0.073	1.11(2)	1.11(2)
0.1425	0.1432	0.065	1.12(2)	1.12(2)
0.1432	0.1432	0.056	1.12(2)	1.12(2)

7.4 WI determination of Z_V and Z_A with the Clover action and heavy quark masses

In this subsection we present the Clover results for the RC's Z_V and Z_A obtained from a direct evaluation of the axial lattice chiral WI. This calculation has been performed by using the set of gauge configurations and quark propagators of Subsect. 7.3.

A determination of Z_V can be obtained by considering the axial WI of (76). In the Clover case, the right-hand side of this equation must be corrected for an $\mathcal{O}(a)$ term as shown in (88), so that the resulting WI has the form:

$$2\rho^{12}\int dx\int d\mathbf{y}\langle P^{12}(x)A_0^{21}(y)V_0^{22}(0)\rangle = -(\frac{1}{Z_V^{22}}-a\rho^{12})\int d\mathbf{y}\langle A_0^{21}(y)A_0^{12}(0)\rangle \quad (132)$$

By computing ρ from (64) (with improved operators and symmetric derivative $\overline{\nabla}^0$) and the two correlation functions appearing in the above WI, we can solve for Z_V.

The KLM improvement to (132) involves both the standard KLM factors, which correct the two- and three-point correlation functions, and the KLM correction to the ratio ρ, given by (121). The results for Z_V obtained from this WI are then presented in Table 9, with and without the KLM corrections. For a comparison, we also show in the table the values of Z_V corresponding to the ratio R_0^{KLM}, which have already been given in Table 7.

Let us first consider in Table 9 the values of Z_V obtained from the WI (132) without the KLM corrections. As the bare quark mass increases from $am \simeq 0$ up to $am \simeq 0.9$ we find that Z_V decreases by approximately 25% due to finite cutoff effects. This is to be contrasted to the increase by about 25% of the R_0 estimate of Z_V (sixth column of the table) in the same mass interval. The two determinations only agree at very small values of the quark mass.

When we apply the KLM prescription we see from the fifth column of Table 9 a dramatic change in the estimates of Z_V. This effect is mainly due to the different normalization of the ratio ρ, and signals the presence of large $\mathcal{O}(a^2m^2)$ corrections. However, in spite of the KLM correction, the variation of Z_V with the quark mass is still of about 25%. Comparing the results of the last two columns of the table, we note a good agreement between the KLM corrected Z_V estimates obtained with the WI and Ratio methods. This suggests that the KLM correction has adjusted the $\mathcal{O}(a^2)$ terms in such a way that the resulting values of the RC's exhibit a more universal behaviour, approximately independent of the specific correlation functions used for their computation.

Fitting Z_V as in (125), we obtain an estimate of the RC's with presumably negligible $\mathcal{O}(am)$ discretization effects. We quote:

$$Z_V^{eff} = 0.80(2) \quad , \quad \text{from WI} \qquad (133)$$

in perfect agreement with the previous determinations from the Ratio method ((129) and (131)).

The RC of the local axial current Z_A, can be computed from the WI of (88), by choosing suitable spatial Lorentz indices of the currents: $\nu = \rho = 1, 2, 3$. Also in this case we find it convenient to take the flavour -2 and -3 quarks degenerate in mass, so that the WI only involves the RC's Z_A^{12} and Z_V^{22}. For the latter, we use the values obtained from the ratio R_0 (at the corresponding quark masses), which is affected by a smaller statistical error, compared to the determination coming from the axial WI. Also for Z_A we have considered the results obtained with and without the KLM prescription, which involves in this case corrections to the correlation functions, to the ratio ρ and to Z_V itself. The resulting values of Z_A are presented in Table 10 and are plotted in Fig. 5 as a function of the effective mass $am_{eff} = (am_1 + am_2)/2$.

As in the case of the vector current, we find from Table 10 and Fig. 5 a large dependence of Z_A on the two quark masses. For values of the effective quark mass of the order of $am \simeq 0.5$ such a dependence can affect the value of the RC by approximately 20%. When we fit Z_A as in (125), we find for the coefficient of the linear term the value $B \simeq -0.13$. This value is of opposite sign but similar in magnitude to the coefficient found for Z_V. This indicates the presence of large $\mathcal{O}(g^2am)$ corrections in both cases. From Fig. 5 we also notice that points corresponding to degenerate and non degenerate quark masses in the current do not lie on the same curve, indicating the presence of a large dependence on the difference of the two quark masses. From the fit we obtain $D \simeq -0.17$ for both uncorrected and KLM corrected data. Therefore, we find

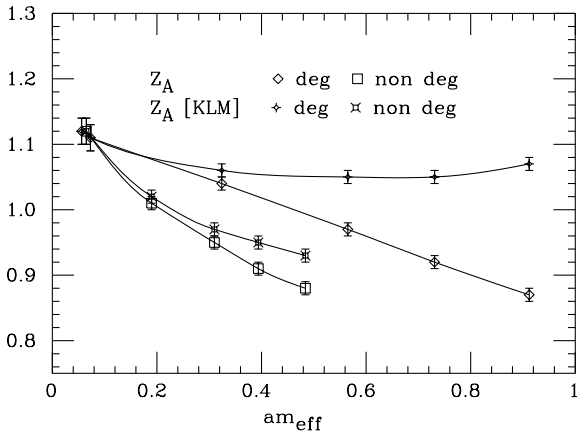

Fig. 5. The RC Z_A for the Clover action, with and without KLM corrections, as a function of the effective quark mass. Lines are only guides to the eye

that the KLM corrections do not improve the finite cutoff effects observed in the numerical estimates of Z_A.

From the zeroth order coefficient in the fit of Z_A we also derive an estimate of the value of the RC with presumably negligible $\mathcal{O}(am)$ effects. We quote:

$$Z_A^{eff} = 1.10(2) \tag{134}$$

in good agreement with the independent determination at light quark masses, presented in Table 6.

8 Further improvements and future outlook

The results presented in this paper lead to the following conclusions:

- Applying various recipes of BPT gives RC's which are consistent to each other within 10% and which differ from those obtained from SPT by at most 20%. In most (but not all) cases, perturbative improvements based on BPT can lead to more reliable estimates of the RC's, in the sense that they are in better agreement with those obtained non-perturbatively.
- At present values of the lattice cutoff, non-perturbative estimates of the RC's based on lattice WI's, suffer, in the Wilson case, from large discretization errors of about 20%. When tree-level Clover improvement is implemented, such systematic effects decrease significantly for correlation functions computed at small quark mass. They remain large, however, as the quark mass increases ($am \gtrsim 0.2$). We have checked that this situation persists for various choices of the field rotation, characteristic of tree-level $\mathcal{O}(a)$ improvement (i.e. different choices of z in (84)).
- A very systematic analysis of the KLM corrections has been carried out, both with Wilson and Clover actions. In the Wilson case, the KLM factors sometimes succeed and sometimes fail to correct large $\mathcal{O}(am)$ effects, depending on the correlation function examined. Moreover, even in the range of small quark masses, large residual $\mathcal{O}(a)$ errors remain, as can be seen from the discrepancies in the Z_V estimates from different correlation functions. In the Clover case, KLM cannot correct the observed large $\mathcal{O}(g_0^2 am)$ systematic errors.

The implication of these results is that, with present day lattice cutoffs of about $2-4$ GeV ($\beta = 6.0-6.4$), simulations of QCD phenomenology in the charm region (e.g. studies of f_D or D-meson semileptonic decays) suffer from sizeable finite cutoff errors. This limits the predictability of lattice QCD for heavy flavour physics. In order to circumvent this problem, and given the impracticability of performing the simulations at significantly smaller lattice spacings, one must seek to improve the lattice convergence to the continuum limit. In this work, we have examined two such efforts, namely Clover tree-level improvement and KLM matching. Here we comment briefly on the recently proposed extensions of these efforts, [32] and [39].

The aim of [32] is non-perturbative $\mathcal{O}(a)$ improvement. To achieve this, the Wilson action must be modified by adding the (dimension-5) Clover term of (77) with the coefficient c_{SW} determined non-perturbatively. In addition, a dimension-d renormalized operator must be cast in the form

$$\hat{O} = Z_O(g^2, a\mu)[1 + b_O(g^2)am] \times [O + a\sum_n c_n(g^2)\delta O_n] \tag{135}$$

where the δO_n's are the dimension-$(d+1)$ operators which are allowed to mix with O. The constants b_O and c_n can either be determined in PT or non-perturbatively. In PT they are known at tree-level and, in some cases, at 1-loop [56]. For example, for bilinear local quark operators, $c_n = 0$ and $b_0 = 1$ correspond to the tree-level Clover improvement discussed in this work, with field rotations determined at $z = 1$. For a full $\mathcal{O}(a)$ improvement, all the improving coefficients must be computed non-perturbatively. In [32] the coefficients c_{SW} of the Clover term and c_A of the axial current have been computed non-perturbatively, with the aid of lattice chiral WI's; for the coefficient c_V of the vector current see [57]. As shown in [34], chiral WI's cannot be used in the spirit of [32] for the determination of the coefficients b_O (with the exception of b_V of the vector current computed in [33]). Two methods for the computation of c_O and b_O, which achieve $\mathcal{O}(a)$ improvement at non-zero quark masses, can be found in [34] and [35]. In this way, the approach to the continuum limit is accelerated to $\mathcal{O}(a^2)$.

The proposal of [39] can be summarized as follows. We saw that the KLM matching consists in the redefinition

$$\hat{O} = Z_O(g_0^2, a\mu)F_O(am)O \tag{136}$$

The KLM factor F_O has been worked out at zero spatial momentum in the free theory and for any value of the quark mass. The authors of [39] propose to extend this calculation beyond the free theory. They claim that it is possible to perform the lattice to continuum matching, again at zero spatial momentum but for any value of the

quark mass, order by order in PT and to all orders in am. However, this matching has not yet been worked out beyond tree-level.

We believe that the non-perturbative Clover improvement of on-shell ME's, proposed in [32], is the most promising way of controlling discretization errors at present day values of the UV cutoff. Our analysis of the tree-level Clover improvement at large quark masses strongly suggests that it is crucial to implement the extension of this program beyond the chiral limit [34,35]. This will hopefully render lattice computations of heavy flavour physics extremely accurate.

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Appendix: The calculation of q^*

We give here the essential details and discuss some subtleties concerning the calculation of q^* for the RC's Z_V, Z_A and the ratio Z_S/Z_P. For definitiveness, we consider the $\overline{\rm MS}$ renormalization prescription in the NDR scheme. The general 1-loop expression of (26), for the three quantities of interest, becomes

$$\begin{aligned} Z_V &= 1 - \frac{g_0(a)^2}{(4\pi)^2} C_F \left[C_V^{LAT} + C_\Sigma^{LAT} \right] \\ Z_A &= 1 - \frac{g_0(a)^2}{(4\pi)^2} C_F \left[C_A^{LAT} + C_\Sigma^{LAT} \right] \\ \frac{Z_S}{Z_P} &= 1 - \frac{g_0(a)^2}{(4\pi)^2} C_F \left[C_S^{LAT} - C_P^{LAT} \right] \end{aligned} \tag{137}$$

These expressions are easily derived from the discussion preceeding Subsect. 3.1. For convenience, we have rescaled the lattice constants C^{LAT} by the colour factor $C_F = 4/3$, which now appears explicitly. Note that none of the above quantities depends on the renormalization scale $a\mu$. Consequently, the q^*'s of these quantities are also independent of $a\mu$.

For the Wilson action, the terms on the r.h.s of the above expressions have been worked out in [2,3]:

$$\begin{aligned} C_V^{LAT} + C_\Sigma^{LAT} &= 1 - \left[\frac{1}{4} I_1 - \frac{1}{2} I_2 - I_3 + I_\Sigma \right] \\ C_A^{LAT} + C_\Sigma^{LAT} &= 1 - \left[\frac{1}{4} I_1 + \frac{1}{2} I_2 - I_3 + I_\Sigma \right] \\ C_S^{LAT} - C_P^{LAT} &= -2 I_2 \end{aligned} \tag{138}$$

where the finite integrals I_1, I_2, I_3 and I_Σ are reproduced here for completeness:

$$\begin{aligned} I_1 &= 4\pi^2 \int_{-\pi}^{\pi} \frac{d^4q}{(2\pi)^4} \left[\frac{\frac{4}{3}(\Delta_1\Delta_4 - \Delta_5) + 2r^2\Delta_4^2 + 4r^4\Delta_1^2\Delta_4}{\Delta_1\Delta_2^2} \right. \\ &\quad \left. + \frac{4\Delta_4(\Delta_3^2 - 2\Delta_1\Delta_3)}{\Delta_1^3\Delta_2^2} \right] \\ I_2 &= 4\pi^2 r^2 \int_{-\pi}^{\pi} \frac{d^4q}{(2\pi)^4} \left[\frac{4\Delta_1(4 - \Delta_1) - \Delta_4}{\Delta_2^2} \right] \\ I_3 &= 4\pi^2 \int_{-\pi}^{\pi} \frac{d^4q}{(2\pi)^4} \left[\frac{-4\Delta_5}{3\Delta_1\Delta_2^2} + \frac{\frac{1}{3}\Delta_4 + 2r^2\Delta_4 + 4r^4\Delta_1^2}{\Delta_2^2} \right] \\ I_\Sigma &= 4\pi^2 \int_{-\pi}^{\pi} \frac{d^4q}{(2\pi)^4} \left[\frac{\Delta_4}{4\Delta_1^3\Delta_2} (\Delta_3 - r^2\Delta_1^2) + \frac{\Delta_5}{4\Delta_1^2\Delta_2} \right. \\ &\quad \left. - \frac{r^2\Delta_6}{\Delta_2} - \frac{\Delta_1}{2} \right] \end{aligned} \tag{139}$$

with

$$\begin{aligned} \Delta_1 &= \sum_\mu \sin^2(q_\mu/2) \\ \Delta_2 &= \sum_\mu \sin^2(q_\mu) + 4r^2 \left[\sum_\mu \sin^2(q_\mu/2) \right]^2 \\ \Delta_3 &= \sum_\mu \sin^4(q_\mu/2) \\ \Delta_4 &= \sum_\mu \sin^2(q_\mu) \\ \Delta_5 &= \sum_\mu \sin^2(q_\mu) \sin^2(q_\mu/2) \\ \Delta_6 &= \sum_\mu \cos(q_\mu) \end{aligned} \tag{140}$$

In the present work, the Wilson parameter has been set to unity; $r = 1$.

The recipe for finding q^* is most straightforward for the ratio Z_S/Z_P. In order to implement (31), we must simply compute the 4-dimensional integral I_2 and also the corresponding integral L_2, obtained by multiplying the integrand of I_2 by a $\ln(q^2)$. L_2 is also finite, and therefore calculable. Thus, for Z_S/Z_P we have

$$\ln(q^{*2}) = \frac{L_2}{I_2} \tag{141}$$

We note in passing that the recipe of (31) is not univocal, even for this simple case: the internal momentum q of the integral I_2, being a dummy variable, can be chosen with a certain degree of arbitrariness (in particular, translations of q) which does not affect the final answer, since the integrand is made up of translationally invariant propagators. However, such changes in q will affect the logarithm in the integrand of L_2, and thus will give us a different q^*. This is only an academic problem, because q^* is only meant to be a typical scale of the process under examination. Thus, estimates of q^* which differ by the choice of internal momenta q are all equally acceptable.

The calculation of q^* for Z_V and Z_A is more complicated. Besides L_2, we also need L_1 and L_3 (again obtained from I_1 and I_3, by modifying their integrands by $\ln(q^2)$). L_1 and L_3 are also finite, so this causes no serious problem. The real complication arises from the unity on the r.h.s

of (138). This constant is obtained as a difference of two integrals which diverge logarithmically with the external momentum a^2p^2. We write them out explicitly. The first one arises from the divergent part of the projected amputated Green functions $\Gamma_V(p)$ and $\Gamma_A(p)$:

$$\hat{I}_\Gamma(ap) = 16\pi^2 \int_{-\pi}^{\pi} \times \frac{d^4q}{(2\pi)^4} \frac{\sum_\rho \sin^2 q_\rho}{\left[4\sum_\mu \sin^2\left(\frac{q_\mu}{2}\right)\right]^2 \left[4\sum_\nu \sin^2\left(\frac{q_\nu + ap_\nu}{2}\right)\right]} = -\ln(a^2p^2) - \gamma_E + F_{0001} + 5/2 + \mathcal{O}(ap) \quad (142)$$

whereas the second integral arises from the divergent part of the self-energy:

$$\hat{I}_\Sigma(ap) = -16\pi^2 \int_{-\pi}^{\pi} \times \frac{d^4q}{(2\pi)^4} \frac{\sum_\rho \sin(q_\rho + ap_\rho)\sin(q_\rho)}{\left[4\sum_\mu \sin^2\left(\frac{q_\mu}{2}\right)\right]^2 \left[4\sum_\nu \sin^2\left(\frac{q_\nu + ap_\nu}{2}\right)\right]} = \ln(a^2p^2) + \gamma_E - F_{0001} - 3/2 + \mathcal{O}(ap) \quad (143)$$

The integration (in the limit $ap \to 0$) has been performed with the aid of the tabulated integrals of [58]. Thus, the factor +1 on the r.h.s. of the first two (138) is obtained from the integral $\hat{I}_{SUM} = \hat{I}_\Gamma + \hat{I}_\Sigma$:

$$\hat{I}_{SUM} = 16\pi^2 \int_{-\pi}^{\pi} \times \frac{d^4q}{(2\pi)^4} \frac{\sum_\rho [\sin(q_\rho) - \sin(q_\rho + ap_\rho)]\sin(q_\rho)}{\left[4\sum_\mu \sin^2\left(\frac{q_\mu}{2}\right)\right]^2 \left[4\sum_\nu \sin^2\left(\frac{q_\nu + ap_\nu}{2}\right)\right]} = 1 + \mathcal{O}(ap) \quad (144)$$

The above integral is finite. Its contribution to the RC's is given by its limit $ap \to 0$. We note that, in this limit the integrand of $\hat{I}_{SUM}$ vanishes, except for $q_\mu = 0$. Thus, the integral is dominated by the region of small q. This enables us to substitute the integrand by its small q and ap limit and take it over the whole q range:

$$\hat{I}_{SUM} = -\frac{1}{\pi^2} \int_{-\infty}^{+\infty} d^4q \frac{ap \cdot q}{q^4(q + ap)^2} \quad (145)$$

We introduce the standard Feynman parameters, and after some trivial algebraic manipulations obtain:

$$\hat{I}_{SUM} = -\frac{2}{\pi^2} \int_0^1 dx(1-x) \times \int_{-\infty}^{+\infty} d^4q \frac{ap \cdot q}{[(q + xap)^2 + (ap)^2 x(1-x)]^3} \quad (146)$$

We next shift the integration variable $q \to q + xap$ and pass over to polar coordinates in q-space. Integrating the polar angles leaves us with

$$\hat{I}_{SUM} = 2(ap)^2 \int_0^1 dx(1-x) \times \int_0^{+\infty} dq^2 \frac{q^2}{[q^2 + (ap)^2 x(1-x)]^3} \quad (147)$$

Finally, we perform a further change of variables $q^2 \to q^2/[a^2p^2x(1-x)]$, and carry out the integration in x to obtain

$$\hat{I}_{SUM} = 2\int_0^{+\infty} dq^2 \frac{q^2}{(q^2+1)^3} \quad (148)$$

The above is easily seen to be equal to 1, which is the final result of (144).

Now the application of (31) requires the introduction of $\ln(q^2)$ in the integrand of $\hat{I}_{SUM}$. However, it is not obvious at which stage of the successive integrations from (144) to (148) this should be done. For example, if we were to introduce $\ln(q^2)$ in (147), we would find a result proportional to $\ln(a^2p^2)$. This does not make sense, since it yields a q^* which depends on the momenta of the external states and, even worse, this ap dependence is singular. Thus we prefer to introduce the $\ln(q^2)$ in the integrand of (148):

$$L_{SUM} = 2\int_0^{+\infty} dq^2 \frac{q^2 \ln(q^2)}{(q^2+1)^3} \quad (149)$$

The above integration can be easily performed; we find $L_{SUM} = 1$, which is a p-independent constant. Thus, q^* for Z_V is given by

$$\ln(q^{*2}) = \frac{1 - \left[\frac{1}{4}L_1 - \frac{1}{2}L_2 - L_3 + L_\Sigma\right]}{1 - \left[\frac{1}{4}I_1 - \frac{1}{2}I_2 - I_3 + I_\Sigma\right]} \quad (150)$$

For Z_A we have

$$\ln(q^{*2}) = \frac{1 - \left[\frac{1}{4}L_1 + \frac{1}{2}L_2 - L_3 + L_\Sigma\right]}{1 - \left[\frac{1}{4}I_1 + \frac{1}{2}I_2 - I_3 + I_\Sigma\right]} \quad (151)$$

The Clover case presents no extra conceptual difficulties. Everything is like in the Wilson case, except that on the r.h.s. of (138) we have more finite integrals, which arise from the extra (Clover) term in the action and the $\not{D}$ field rotations of Subsect. 5.2. These finite integrals are listed in [7,8].

A final comment is in place here. A shortcut to all the subtleties related to L_{SUM} would be to simply ignore the contribution of this integral to q^*. In this case, q^* is obtained from the last two equations, with the +1's omitted from both numerator and denominator. With this q^* the most affected RC's are Z_V and Z_A of the Clover action. Their values change by less than 2%, for all boosting recipes of the coupling.

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