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Anomalous three gauge boson couplings in $e^+e^- \to W^+W^-$ and "optimal" strategies for their measurement

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Abstract. We discuss how to measure anomalous WWZ- and $WW\gamma$-couplings with minimal statistical error using integrated observables, without having to assume that the anomalous couplings are small. We propose a parametrisation of these couplings which is well suited for the extraction of both single and many parameters, and which leads to a very simple form of the integrated cross section, from which additional information on the couplings can be obtained.

1 Introduction

The direct and precise measurement of the self-coupling between the electroweak gauge bosons in W-pair production will be a crucial step in testing the standard model of electroweak interactions and searching for physics beyond it. It will form an important part of the physics programme at LEP2 and at a planned linear e^+e^--collider (LC). As is well known there are three diagrams at tree level that contribute to the amplitude of $e^+e^- \to W^+W^-$ in the standard model, one with t-channel neutrino exchange and the other two with a γ or Z in the s-channel, involving the vertices $WW\gamma$ and WWZ. One can parametrise the corresponding vertex functions in order to quantify the couplings and to compare them with their form in the standard model. Requiring Lorentz covariance each vertex is described by seven complex form factors [1], three of which give couplings that violate CP symmetry. This parametrisation neglects the scalar components of the vector bosons, which do not contribute in scattering amplitudes where the bosons are on shell or couple to massless fermions.

Without further physical assumptions one is thus left with 28 real parameters whose simultaneous extraction in one experiment looks quite hopeless. Given the limited event statistics expected at both LEP2 and the LC one will only obtain meaningful errors on a reduced number of coupling parameters at one time. This may be achieved by imposing certain constraints on the full set of coupling constants; various suggestions for such constraints based on symmetry considerations have been made in the literature [2,3]. One must however keep in mind that experimental values or bounds on couplings that have been obtained with particular constraints cannot be converted into results without constraints or with different ones; the information lost by assuming relations between couplings cannot be retrieved. Although imposing such constraints is certainly legitimate and can be useful we stress that a data analysis with independent couplings will be valuable, both to decrease model dependence and to allow comparison of results from different experiments.

We remark that of course one can also give (reasonably small) errors on *single or few* couplings in a multiparameter analysis. In this paper we propose a parametrisation of the couplings which is well adapted to this end, the statistical errors on the different measured parameters being approximately uncorrelated.

We will work in the framework of optimal observables, a way to extract unknown coupling parameters introduced for the case of one parameter in [4,5] that has since been used for various reactions [6–8]. General aspects of this method, in particular its extension to an arbitrary number of parameters, as well as its application to W^+W^- production were discussed in [9]. In this paper we investigate again the reaction $e^+e^- \to W^+W^-$. We concentrate here on the decay channels, where one W decays hadronically and the other into an electron or muon and its neutrino. Calculated with the Born level cross section of the standard model the statistics of these channels is about 3000 events for a collision energy of $\sqrt{s} = 190$ GeV and 500 pb^{-1} integrated luminosity, which are typical planned

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LEP2 parameters, and about 22000 events with 10 fb^{-1} at $\sqrt{s} = 500$ GeV, which might be achieved at the LC.

A complementary source of information is the integrated cross section, which is a quadratic function of the triple gauge couplings. The combination of information from the total event rate and from observables that make use of the detailed distribution in the final state has for example been used in [10], where *CP* violation in the decay $Z \to b\bar{b}g$ was investigated.

In Sect. 2 of this paper we will further develop some aspects of the method of optimal observables, in particular we will discuss what can and what cannot be achieved with this method in the general case, i.e. without the linear approximation in the coupling parameters that was used in [9]. In Sect. 3 we then propose a parametrisation of the couplings that simultaneously diagonalises certain matrices connected with our observables and with the integrated cross section. These parameters achieve two goals: their quadratic contribution to the total cross section is a simple sum of squares and the covariance matrix of the corresponding optimal observables is diagonal. The methods which we use for this purpose are borrowed from the theory of small oscillations of a system with f degrees of freedom (cf. e.g. [11]). Our parameters correspond to "normal coordinates" and their use in an experimental analysis should in our view present several advantages. We give some numerical examples for W-pair production at LEP2 and the LC in Sect. 4 and make some further remarks on how our proposal might be implemented in practice in Sect. 5. The last section of this paper gives a summary of our main points.

2 Optimal observables: analysis beyond leading order

The method of optimal observables has previously been presented in the approximation that the couplings to be extracted are sufficiently small to allow for a leading order Taylor expansion of various expressions. Here we show how to use it beyond this approximation.

Let us denote by g_i the real and imaginary parts of the $WW\gamma$ and WWZ form factors minus their values in the standard model at tree level. As the amplitude of our process is linear in these couplings we can write the differential cross section as

$$\frac{d\sigma}{d\phi} = S_0(\phi) + \sum_i S_{1,i}(\phi)\, g_i + \sum_{ij} S_{2,ij}(\phi)\, g_i g_j \quad , \qquad (1)$$

where $S_{2,ij}(\phi)$ is a positive semidefinite symmetric matrix and ϕ collectively denotes the set of measured phase space variables. We assume here that nonstandard effects in our reaction can be described in terms of the above $WW\gamma$ and WWZ couplings, so that S_0, $S_{1,i}$ and $S_{2,ij}$ are known functions of ϕ. A general discussion of W^+W^- production in e^+e^- collisions has been given in [12], and in four fermion production there can yet be other effects beyond the standard model.

The integrated cross section is

$$\sigma = \sigma_0 \left(1 + \sum_i \hat{\sigma}_{1,i}\, g_i + \sum_{ij} \hat{\sigma}_{2,ij}\, g_i g_j \right) \qquad (2)$$

with the standard model cross section $\sigma_0 = \int d\phi\, S_0(\phi)$ and coefficients

$$\hat{\sigma}_{1,i} = \frac{1}{\sigma_0} \int d\phi\, S_{1,i}(\phi) \quad , \quad \hat{\sigma}_{2,ij} = \frac{1}{\sigma_0} \int d\phi\, S_{2,ij}(\phi) \quad . \qquad (3)$$

The idea of using integrated observables is to define suitable functions $\mathcal{O}_i(\phi)$ of the phase space variables and to extract the unknown couplings from their measured mean values $\overline{\mathcal{O}}_i$. Let us give the details. From (1) and (2) we obtain the expectation value $E[\mathcal{O}_i]$ of $\mathcal{O}_i$ as

$$E[\mathcal{O}_i] - E_0[\mathcal{O}_i] = \frac{\displaystyle\sum_j c_{ij}\, g_j + \sum_{jk} q_{ijk}\, g_j g_k}{\displaystyle 1 + \sum_j \hat{\sigma}_{1,j}\, g_j + \sum_{jk} \hat{\sigma}_{2,jk}\, g_j g_k} \qquad (4)$$

with the standard model expectation value $E_0[\mathcal{O}_i] = (\int d\phi\, \mathcal{O}_i S_0)/\sigma_0$ and coefficients

$$\begin{aligned} c_{ij} &= \frac{1}{\sigma_0} \int d\phi\, \mathcal{O}_i S_{1,j} - E_0[\mathcal{O}_i]\, \hat{\sigma}_{1,j} \quad , \\ q_{ijk} &= \frac{1}{\sigma_0} \int d\phi\, \mathcal{O}_i S_{2,jk} - E_0[\mathcal{O}_i]\, \hat{\sigma}_{2,jk} \quad . \end{aligned} \qquad (5)$$

We remark in passing that the coefficients in (4) can be written in a compact form as

$$\begin{aligned} c_{ij} &= V_0[\mathcal{O}_i\,,\, S_{1,j}/S_0] \quad , \quad q_{ijk} &&= V_0[\mathcal{O}_i\,,\, S_{2,jk}/S_0] \quad , \\ \hat{\sigma}_{1,j} &= E_0[S_{1,j}/S_0] \quad , \quad \hat{\sigma}_{2,jk} &&= E_0[S_{2,jk}/S_0] \quad , \end{aligned} \qquad (6)$$

where $V_0[f,g] = E_0[fg] - E_0[f]\, E_0[g]$ is the covariance of $f(\phi)$ and $g(\phi)$ in the standard model. Note that $\hat{\sigma}_{2,jk}$ is symmetric and positive definite, whereas q_{ijk} as a matrix in j and k is symmetric but in general indefinite.

An estimation of the couplings can now be obtained by solving the system (4) with $E[\mathcal{O}_i]$ replaced by the mean values $\overline{\mathcal{O}}_i$,

$$\overline{\mathcal{O}}_i - E_0[\mathcal{O}_i] = \frac{\displaystyle\sum_j c_{ij}\, g_j + \sum_{jk} q_{ijk}\, g_j g_k}{\displaystyle 1 + \sum_j \hat{\sigma}_{1,j}\, g_j + \sum_{jk} \hat{\sigma}_{2,jk}\, g_j g_k} \quad , \qquad (7)$$

provided of course one has n observables for n unknown couplings. When the system (7) is linearised in the g_i it is easily solved by inversion of the matrix c_{ij}. One is however not constrained to do so and can instead solve the exact set of equations (7). By multiplication with the denominator it can be rearranged to a coupled set of quadratic equations in the g_i and will in general have several solutions. Some of these may be complex and thus ruled out, but from the information of the $\overline{\mathcal{O}}_i$ alone one cannot tell

which of the remaining real ones is the physical solution. We will come back to this point.

The measured mean values $\overline{\mathcal{O}}_i$ are of course only equal to the $E[\mathcal{O}_i]$ up to systematic and statistical errors. We only consider the latter here, which are given by the covariance matrix $V(\mathcal{O})_{ij}$ of the observables $\mathcal{O}_i$ divided by the number N of events in the analysis. To convert the errors on the observables into errors on the extracted couplings we use the quantity

$$\chi^2(g) = \sum_{ij} \left(\overline{\mathcal{O}}_i - E[\mathcal{O}_i]\right) N V(\mathcal{O})^{-1}{}_{ij} \left(\overline{\mathcal{O}}_j - E[\mathcal{O}_j]\right) \quad , \tag{8}$$

which depends on the g_i through the $E[\mathcal{O}_i]$ given in (4). Solving (7) is tantamount to minimising χ^2 with $\chi^2_{min} = 0$, and a confidence region on the couplings is as usual given by

$$\chi^2(g) - \chi^2_{min} \leq const. \tag{9}$$

with the constant determined by the desired confidence level. In the case of several unknown couplings it might actually be easier to extract the g_i by solving the equation $\chi^2(g) = 0$ using a minimisation procedure, rather than solving the coupled system (7).

There are several possible choices for the covariance matrix $V(\mathcal{O})$ in (8). It can be

1. determined from the measured distribution of the observables $\mathcal{O}_i$,
2. calculated from the differential cross section (1), taking for the g_i the values extracted in the measurement,
3. calculated for vanishing couplings g_i.

Choices 1. and 2. should lead to the same results in the limit of large N where the statistical errors on the measured $V(\mathcal{O})_{ij}$ and g_i become small. Choice 3. in turn will be a good approximation of 2. if the couplings are small enough. If one extracts the couplings by minimising $\chi^2(g)$ one can of course not use choice 2.

Comparison of the covariance matrices obtained by methods 1. and 2. might be helpful to rule out unphysical solutions of (7). The information from the integrated cross section, which we will discuss in Sect. 3.1, can also be used to this effect. Some solutions, however, cannot be eliminated in this way: in [13] it was shown that there are points in the space of couplings which give nearly identical differential cross sections for our process so that unless the statistics are sufficiently large *no* extraction method can determine which of them is the physical solution. The unphysical values of couplings that were extracted in [2, 13] from events generated with all g_i set to zero are indeed obtained as additional solutions of (7) in our method [14].

In [9] we considered an analysis at leading order in the g_i, where one uses the linearised form of (7) to estimate the couplings:

$$\hat{g}_j = \sum_k c^{-1}{}_{jk} \left(\overline{\mathcal{O}}_k - E_0[\mathcal{O}_k]\right) \quad . \tag{10}$$

Correspondingly the linear approximation of (4) is used in the expression (8) of χ^2 which then reads

$$\chi^2(g) = \sum_{ij} (\hat{g}_i - g_i)\, V(g)^{-1}{}_{ij}\, (\hat{g}_j - g_j) \quad , \tag{11}$$

where

$$V(g)^{-1} = N\, c^T \cdot V(\mathcal{O})^{-1} \cdot c \tag{12}$$

is the inverse covariance matrix of the estimated couplings [9]. As one works to leading order in the g_i one can approximate $V(\mathcal{O})$ by its value for zero couplings, i.e. choose possibility 3. above.

The confidence regions $\chi^2(g) \leq const.$ for the measured couplings are then ellipsoids in the space of the g_i with centre at $(\hat{g}_i)$. The optimal observables

$$\mathcal{O}_i(\phi) = \frac{S_{1,i}(\phi)}{S_0(\phi)} \tag{13}$$

discussed in [9] have the property that to leading order the statistical errors on the estimated couplings are the smallest possible ones that can be obtained with *any* method, including e.g. a maximum likelihood fit to the normalised distribution of ϕ obtained from the differential cross section (1).

Note that one can still use the linearised expressions (10) and (11) in an analysis beyond leading order. The error $\chi^2(g) \leq 1$ on the couplings will be given by an ellipsoid with defining matrix (12), where $V(\mathcal{O})$ is the covariance matrix at the actual values of the couplings. These errors will in general no longer be optimal, so that when the leading order approximation is not good one might obtain better errors with a different choice of observables. More importantly, however, the extracted values of the couplings are biased: averaged over a large number of experiments the measured couplings differ from the actual ones by terms quadratic in the g_i.

If instead one uses the full expressions (4), (7) and (8) one has no bias on the extracted coupling parameters, provided the number N of events in the analysis is large enough. Let us see if we can find optimal observables for this case. To this end we expand the differential cross section around some values $\tilde{g}_i$ of the couplings:

$$\begin{aligned} \frac{d\sigma}{d\phi} = \widetilde{S}_0 &+ \sum_i \widetilde{S}_{1,i}(\phi) \cdot (g_i - \tilde{g}_i) \\ &+ \sum_{ij} \widetilde{S}_{2,ij}(\phi) \cdot (g_i - \tilde{g}_i)(g_j - \tilde{g}_j) \quad . \end{aligned} \tag{14}$$

The corresponding zeroth order cross sections and mean values are $\widetilde{\sigma}_0 = \int d\phi\, \widetilde{S}_0$ and $\widetilde{E}_0[\mathcal{O}_i] = (\int d\phi\, \mathcal{O}_i \widetilde{S}_0)/\widetilde{\sigma}_0$, respectively. We then can re-express $E[\mathcal{O}_i]$ in (4), replacing g_i with $g_i - \tilde{g}_i$, E_0 with $\widetilde{E}_0$, and using new coefficients $\tilde{c}_{ij}$ etc. constructed as in (3), (5). Making the same replacements in (7) we have an alternative set of equations to extract the coupling parameters.

It can be shown that for sufficiently large N the confidence regions obtained from (8), (9) in a nonlinear analysis are again ellipsoids given by $\chi^2(g) \leq const.$ One can then write $\chi^2(g)$ as in (11), but with $V(g)^{-1}$ of (12) replaced by

$$V(g)^{-1} = N\, \tilde{c}^{\,T} \cdot V(\mathcal{O})^{-1} \cdot \tilde{c} \quad , \tag{15}$$

where $\tilde{c}$ corresponds to an expansion (14) of $d\sigma/d\phi$ about the *actual* values of the couplings. The main point of the

argument is that for large N the statistical errors on the $\overline{\mathcal{O}}_i$ become small, so that the extracted couplings will be sufficiently close to the actual ones to allow for a linearisation of (4) and (7), cf. [15–17].

Finally one can construct new observables

$$\widetilde{\mathcal{O}}_i(\phi) = \frac{\widetilde{S}_{1,i}(\phi)}{\widetilde{S}_0(\phi)} \tag{16}$$

from (14). They will be optimal, i.e. have minimum statistical error if the $\tilde{g}_i$ are equal to the actual values of the g_i. In appendix A we show that, up to linear reparametrisations given in (39), this is the only set of n integrated observables that measures the n couplings with minimum error. There is hence no choice of observables that would be optimal for *all* values of the actual coupling parameters. As these are unknown one can in practice not write down the truly "optimal" observables, but our argument tells us how one can improve on the choice in (13) if one has some previous estimates $\tilde{g}_i$ of the couplings (cf. also [5]). One may then choose to perform a leading order analysis as described above, linearising about $g_i = \tilde{g}_i$ instead of $g_i = 0$. A practical way to proceed could be to estimate the parameters g_i at first using the linearised method around $g_i = 0$. Suppose this gives as best estimate some values $\hat{g}_i$. Then in a second step one could set $\tilde{g}_i = \hat{g}_i$ and use the linearised method around $\tilde{g}_i$ to improve the estimate etc.

At this point we wish to comment on the "optimal technique" for determining unknown parameters in the differential cross section that has been proposed in [18]. The "weighting functions" $w_i(\phi)$ there depend on the actual values of the parameters one wants to extract and are thus not "observables". Only if one sets the unknown parameters in the $w_i(\phi)$ equal to some previous estimates of them can one use these functions to weight individual events; the better these estimates are the more sensitive the functions will be. If one does this then the set $w_i(\phi)$ is equivalent to our observables (16) defined for some estimates $\tilde{g}_i$ of the coupling parameters.

We finally remark that if N is not large enough the statistical errors on the mean values $\overline{\mathcal{O}}_i$ and thus on the measured couplings might be so large that they lead into a region where a linearisation of (4) is not a good approximation. The covariance matrix $V(g)$ is then no longer given by (15). Moreover the errors on the couplings might be asymmetric and the shape of the confidence region defined by (8), (9) very different from an ellipsoid, so that knowledge of $V(g)$ is not sufficient to estimate the errors on the g_i. In such a case we cannot say on general grounds how sensitive our observables are. Incidentally this also holds for other extraction methods such as maximum likelihood fits, whose optimal properties are realised in the limit $N \to \infty$. If one is rather far from this limit the sensitivity of a method will have to be determined by other means, e.g. by detailed Monte Carlo simulations.

The method we have outlined can of course also be applied if one chooses to reduce the number of unknown parameters by imposing certain linear constraints on the couplings. One may still use the observables (13) corresponding to the *full* set of couplings but minimise χ^2 in (8) for the *reduced* set; in this case one can of course not take choice 2. for $V(\mathcal{O})$. In general χ^2_{min} is then different from zero and its value indicates to which extent the particular constraints on the couplings are compatible with the data. If N is large enough χ^2_{min} follows in fact a χ^2-distribution with $n - m$ degrees of freedom for n observables and m independent couplings so that its value can be converted into a confidence level [15].

We conclude with a remark on the use of optimal observables in practice. A realistic data analysis will not be good enough if the Born approximation of the differential cross section (1) is used. Both higher-order theoretical corrections, such as initial state radiation and the finite W width, and experimental effects like detection efficiency and resolution will modify the observed distribution of the phase space parameters ϕ. If they are taken into account in the determination of the coefficients in (4), (7) and of the covariance matrix $V(\mathcal{O})$ they will *not* lead to any bias in the extraction of the couplings and their errors. While this will presumably be done with sets of generated events and might be computationally intensive one still has to determine only a rather limited number of "sensitivity" constants. We remark that the coefficients in the denominator of (7) can be determined separately from the others since they appear in the expression of the integrated cross section (2). On the other hand one needs to know the observables $\mathcal{O}_i(\phi)$ of (13) as functions over the entire experimental phase space, so that the expressions of S_0 and $S_{1,i}$ used to construct them will in practice be taken from a less sophisticated approximation to the actual distribution of ϕ in order to keep them manageable. The observables are then no longer optimal, and it will depend on the individual case which approximations of S_0, $S_{1,i}$ are good enough to obtain observables with a sensitivity close to the optimal one.

2.1 Discrete symmetries

In [9] it was shown how with a suitable combination of all semileptonic W^-W^+ decay channels one can define observables that are either even or odd under the discrete transformations CP and $CP\tilde{T}$, where C denotes charge conjugation, P the parity transformation, and $\tilde{T}$ the "naive" time reversal operation which flips particle momenta and spins but does not interchange initial and final state. Under the conditions on the experimental setup and event selection spelt out in [9] we have two important symmetry properties:

1. A CP odd observable can only have a nonzero expectation value if CP symmetry is violated in the reaction.
2. If the expectation value of a $CP\tilde{T}$ odd observable is nonzero the transition amplitude must have an absorptive part whose phase must satisfy certain requirements in order to give an interference with the nonabsorptive part of the amplitude.

We assume in this analysis that any nonstandard physics in the reaction is due to the triple gauge vertices. In the standard model one needs at least two loops to violate

CP; to a good accuracy the triple gauge couplings are therefore the only possible source of CP violation. For our process, i.e. e^+e^- annihilation into four fermions, an absorptive part that satisfies the requirements mentioned in point 2. will appear in the Standard Model already at next-to-leading order in the electroweak fine structure constant, either through nonresonant diagrams or through loop corrections. To leading order, however, they are only due to the imaginary parts of triple gauge couplings.

In this approximation the optimal observables (13) are CP even (odd) if they correspond to CP conserving (violating) couplings, and $CP\tilde{T}$ even (odd) if they correspond to the real (imaginary) parts of form factors. The coefficient matrix c_{ij} is then block diagonal in four symmetry classes of observables and three-boson-couplings:

(a) : CP and $CP\tilde{T}$ even
(b) : CP even and $CP\tilde{T}$ odd
(c) : CP odd and $CP\tilde{T}$ even
(d) : CP and $CP\tilde{T}$ odd.

In the leading order analysis one thus can treat these four classes of couplings separately and benefit from a great reduction of unknown parameters. Beyond leading order, however, form factors of any symmetry can contribute to $E[\mathcal{O}_i]$:

- in the integrated cross section and thus in the denominator of $E[\mathcal{O}_i] - E_0[\mathcal{O}_i]$ in (4) couplings of all four classes enter quadratically, couplings of class (a) also appear linearly;
- if $\mathcal{O}_i$ belongs to class (a) the numerator of $E[\mathcal{O}_i] - E_0[\mathcal{O}_i]$ has terms linear in the couplings of this class but couplings of all four classes enter quadratically through q_{ijk} ;
- if $\mathcal{O}_i$ belongs to a CP ($CP\tilde{T}$) odd coupling then the numerator in (4) is only linear in CP ($CP\tilde{T}$) odd couplings, but it contains also quadratic terms where a CP ($CP\tilde{T}$) odd coupling is multiplied with a CP ($CP\tilde{T}$) even one.

We remark that this leads to different behaviours of $E[\mathcal{O}_i]$ as one or more couplings g_i become large: whereas for observables in classes (b), (c) and (d) the expectation value goes to zero when a coupling of the same class goes to plus or minus infinity the corresponding limit of an observable in class (a) can be a positive or negative constant or zero.

In a nonlinear analysis one will therefore in principle have to consider couplings with all symmetries at the same time. In practice one might choose simpler procedures if the linear approximation is expected to be not too bad and if one wants to calculate corrections to it. One might for instance first analyse the four symmetry classes separately, neglecting in each case the contributions of the three other classes at the r.h.s. of (7) and then refine the analysis of a class by taking the values obtained in the first step for the couplings in the other classes as fixed in (7).

We emphasise that even beyond the leading order approximation it is still true that a nonzero mean value of a CP or a $CP\tilde{T}$ odd observable is an unambiguous sign of CP violation or the presence of absorptive parts in the process, respectively. The extraction of the values of the couplings, however, becomes more involved than in leading order.

3 Diagonalisation in the couplings

We shall now propose a method to analyse the data which presents several advantages in view of the basic problem posed by the large number of unknown three-boson couplings: with limited event statistics significant error bounds can only be obtained for subsets of the coupling parameters, but imposing constraints on the couplings to reduce their number entails a loss of information that cannot be retrieved. In view of this it should be advantageous to use a parametrisation of the couplings which in a given process and at a given c.m. energy has the following properties:

1. It allows to find observables which are only sensitive to one particular coupling parameter.
2. The induced errors on the couplings determined from these observables are statistically independent.

With this we can on the one hand give single errors for each parameter, on the other hand we can recover from the single errors the multidimensional error of the full set of couplings, having avoided the loss of information incurred by imposing constraints. From the single errors we can also directly see which combinations of couplings in more conventional parametrisations can be measured with good accuracy and to which one is rather insensitive.

Let us remark that in the leading order analysis there is a set of observables satisfying point 1. in *any* parametrisation of the couplings. The linear combinations

$$\mathcal{C}_i = \sum_j {c^{-1}}_{ij}\, \mathcal{O}_j \tag{17}$$

of our optimal observables (13) are only sensitive to g_i for each i (cf. also [18]). The errors on the couplings determined from these observables are, however, in general not uncorrelated; in fact their correlations are the same as those obtained with the original set $\mathcal{O}_i$. This can be seen as follows: going from the $\mathcal{O}_i$ to the $\mathcal{C}_i$ we must replace

$$\begin{aligned} c_{ij} &\to \delta_{ij} \quad , \\ V(\mathcal{O}) \to V(\mathcal{C}) &= c^{-1} \cdot V(\mathcal{O}) \cdot (c^{-1})^T \quad , \end{aligned} \tag{18}$$

so that we have from (12)

$$\begin{aligned} V(g)^{-1}\big|_{\mathcal{O}} &= N\, c^T \cdot V(\mathcal{O})^{-1} \cdot c = N\, V(\mathcal{C})^{-1} \\ &= V(g)^{-1}\big|_{\mathcal{C}} \quad . \end{aligned} \tag{19}$$

In such a case the single errors give an incomplete picture of the situation if correlations are large. This is illustrated in Fig. 1 (a), where the 1–σ ellipsis for two parameters is shown. Their single errors are given by its projection on the coordinate axes and in our example are both rather large. Some linear combinations of them are however measurable with much better precision, which one can only

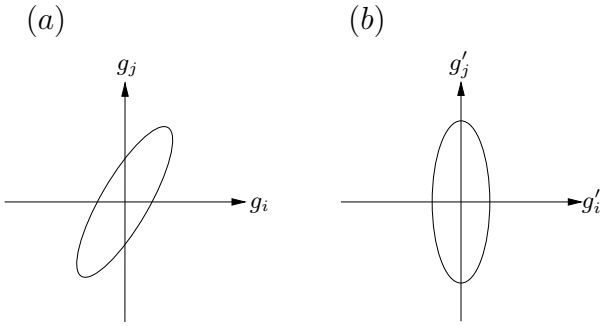

Fig. 1. Example of the 1–σ ellipsis for two extracted parameters in a parametrisation where their errors are correlated **a** or uncorrelated **b**. The single errors on the couplings are given by the projection of the ellipses on the coordinate axes

recognise if both errors and their correlations are given. In Fig. 1 (*b*) where a set of couplings leading to uncorrelated errors is used the situation is much simpler. Note also that the number of correlations, i.e. off-diagonals in $V(g)$, is yet modest for two couplings but increases rapidly with their number.

We will now first see that a parametrisation of the couplings satisfying both points 1. and 2. above can be found in idealised circumstances, and then mention the restrictions one will encounter under more realistic assumptions.

If the statistical errors are small enough to permit an analysis linearised around some couplings $\tilde{g}_i$ the solution to our problem is easily found. For ease of notation we present it taking the $\tilde{g}_i$ to be zero. Starting from the set of couplings g_i and the corresponding optimal observables $\mathcal{O}_i$ in (13) we can go to another set g'_i by

$$\mathbf{g}' = A^{-1}\mathbf{g} \ , \tag{20}$$

where we use vector and matrix notation. The coefficients in the expansion of the differential cross section and the optimal observables transform as follows:

$$\begin{aligned} \mathbf{S}'_1 &= A^T \mathbf{S}_1 \\ \mathcal{O}' &= A^T \mathcal{O} \ . \end{aligned} \tag{21}$$

Let now $\mathcal{O}_i$ be an arbitrary set of observables related to g_i and define the corresponding $\mathcal{O}'_i$ related to g'_i as in (21). Then we have for the matrices relevant for our analysis the following transformation properties:

$$\begin{aligned} c' &= A^T \cdot c \cdot A \\ V(\mathcal{O}') &= A^T \cdot V(\mathcal{O}) \cdot A \\ V(g')^{-1} &= A^T \cdot V(g)^{-1} \cdot A \ . \end{aligned} \tag{22}$$

As shown in [9] our optimal observables satisfy $V(\mathcal{O}) = c$ and $V(g)^{-1} = Nc$ so that for them one can choose a transformation A which diagonalises all three matrices. This new set g' of parameters obviously has the properties 1. and 2. we were looking for.

Beyond the linear approximation of (4) the expectation value of $\mathcal{O}'_i$ will still receive contributions from several couplings. In fact there is no set of observables for which the full nonlinear expression in (4) satisfies point 1. exactly, because the denominator involves quadratic terms in *all* couplings, and this cannot be changed by any linear transformation of the couplings. If on the other hand the statistical errors are too large the covariance matrix $V(g)$ will not give a good picture of the errors as we discussed in Sect. 2, and its diagonalisation will not ensure point 2. In the case however where nonlinear effects in the determination of the couplings and their errors are not too large, i.e. where the leading order expressions are a good first approximation both points 1. and 2. above will still be *approximately* satisfied in a full nonlinear analysis.

We emphasise that if one has some previous estimates $\tilde{g}_i$ of the couplings that considerably deviate from zero one may reduce nonlinear effects in the determination of the g_i by working with an expansion of $d\sigma/d\phi$ around the $\tilde{g}_i$ as shown in Sect. 2; in our diagonalisation programme one will then use couplings $g_i - \tilde{g}_i$ instead of g_i, the matrix $\tilde{c}$ instead of c etc. Thus the usefulness of diagonalisation is *not* restricted to the case that the actual couplings are close to zero. It does, however, require the statistical errors to be sufficiently small for the analysis to stay in a region where linearisation about some values $\tilde{g}_i$ is possible.

To the extent that the observables (13) are constructed from expressions of S_0 and $S_{1,j}$ which are only approximations of those that determine the experimentally observed kinematical distribution the matrices c, $V(\mathcal{O})$ and $N^{-1}V(g)^{-1}$ will not quite be the same and cannot be diagonalised at the same time. One can then diagonalise either $V(\mathcal{O})$ or $V(g)^{-1}$ because they are by definition symmetric and positive definite, whereas c is not necessarily so. Again, unless such effects are large one will end up with a matrix c' that is not diagonal but has relatively small off-diagonals.

It should also be borne in mind that the covariance matrix $V(g)$ only gives the statistical errors on the couplings, so that even if it is exactly diagonal the final errors may be correlated due to systematics.

3.1 Simultaneous diagonalisation of the correlation matrix and the quadratic term in the total cross section

The choice of transformation in (20) to (22) is not unique if one does not require A to be orthogonal.[1] We see in fact no strong argument in favour of an orthogonal transformation and remark that the various parametrisations of the $WW\gamma$ and WWZ couplings in the literature are related by non-orthogonal linear transformations. The freedom to choose A can be used to impose additional conditions on the transformation, and the one we propose here is that the transformed quadratic coefficient

$$\hat{\sigma}'_2 = A^T \cdot \hat{\sigma}_2 \cdot A \tag{23}$$

[1] Orthogonal transformations have been used in the second paper of [7]

in the integrated cross section be the unit matrix. In terms of the new couplings one then has

$$\sigma/\sigma_0 = 1 + \sum_{i=1}^{8} \hat{\sigma}'_{1,i}\, g'_i + \sum_{i=1}^{28} (g'_i)^2 \quad , \tag{24}$$

where we choose the numbering such that g'_1 to g'_8 belong to symmetry class (a) introduced in Sect. 2.1, i.e. they are the CP and $CP\tilde{T}$ even couplings. Only these appear linearly in the cross section, whereas all couplings give a quadratic contribution with coefficient one. Having $\hat{\sigma}'_{2,ij} = \delta_{ij}$ leads to a convenient simplification of (4), (7). Moreover, the measurement of the total cross section gives complementary information on the unknown couplings. Rewriting (24) as

$$\sigma/\sigma_0 = 1 - \sum_{i=1}^{8} \frac{(\hat{\sigma}'_{1,i})^2}{4} + \sum_{i=1}^{8} \left(g'_i + \frac{\hat{\sigma}'_{1,i}}{2} \right)^2 + \sum_{i=9}^{28} (g'_i)^2 \tag{25}$$

we see that measuring a cross section σ_{exp} within an error $\Delta\sigma$ constrains the couplings to be in a shell between two hyperspheres with centre at

$$\begin{aligned} g'_i &= -\frac{\hat{\sigma}'_{1,i}}{2} \quad (i = 1, \ldots, 8) \quad , \\ g'_i &= 0 \quad (i = 9, \ldots, 28) \end{aligned} \tag{26}$$

in the space of all couplings as shown in Fig. 2. Their radii are given by

$$\begin{aligned} r_>^2 &= \frac{\sigma_{exp} + \Delta\sigma - \sigma_{min}}{\sigma_0} \quad , \\ r_<^2 &= \frac{\sigma_{exp} - \Delta\sigma - \sigma_{min}}{\sigma_0} \quad . \end{aligned} \tag{27}$$

Here

$$\sigma_{min} = \sigma_0 \left[1 - \sum_{i=1}^{8} \frac{(\hat{\sigma}'_{1,i})^2}{4} \right] \tag{28}$$

is the smallest value the cross section can attain; that such a minimum exists has been pointed out in [19]. If in (27) $r_>^2$ is positive but $r_<^2$ negative the couplings are inside the hypersphere with radius $r_>$, and if both $r_>^2$ and $r_<^2$ are negative the ansatz (1) for the cross section is inconsistent with the data within the error $\Delta\sigma$.

Such constraints can be useful to find the physical set of couplings when the solution of (7) from the measurement of the optimal observables is not unique. If they are strong enough they might even restrict the couplings to the region where (7) can be linearised and thus simplify their extraction. One can of course use the information from the integrated cross section working with any set of couplings, but again the situation is particularly simple with the form (24).

We note that the information from the total rate is complementary to what is extracted from the mean values of our observables, which involve normalised kinematical distributions. From an experimental point of view their respective measurements will presumably have quite different systematic errors. Let us also recall that the measurement of the mean values $\overline{\mathcal{O}}_i$ times the number N of events obtained with a fixed integrated luminosity combines the information of both [9]. A nonlinear data analysis as presented in Sect. 2 can also be done in this case. We draw attention to the fact that unphysical solutions of equation (7) for $\overline{\mathcal{O}}_i$ and of its analogue for $N\overline{\mathcal{O}}_i$ will in general not be the same. Also the expression of $N\overline{\mathcal{O}}_i$ is a quadratic polynomial in the g_i, so that $\overline{\mathcal{O}}_i$ and $N\overline{\mathcal{O}}_i$ have a different behaviour for large values of the couplings. We shall however not elaborate on these points here.

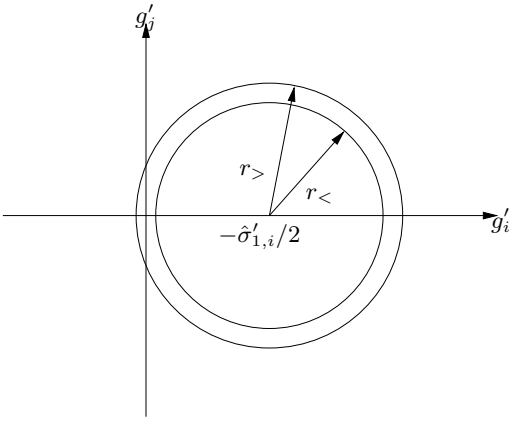

Fig. 2. The measurement of the integrated cross section constrains the couplings to a shell between two hyperspheres with radii $r_>$ and $r_<$ given by (27). g'_i belongs to symmetry class (a) introduced in Sect. 2.1 and g'_j to class (b), (c) or (d). In the example shown here the measurement is compatible with both couplings being zero

Another aspect of the couplings with the property (24) is the following. It is well known that constant coupling parameters deviating from the standard model tree level values lead to amplitudes that violate unitarity [20]. The coefficients $\hat{\sigma}_{1,i}$ and $\hat{\sigma}_{2,ij}$ in the total cross section σ increase strongly with the e^+e^- c.m. energy $\sqrt{s}$ and the couplings g_i must vanish as s becomes large to ensure a decent high-energy behaviour of σ. In our new parametrisation the quadratic coefficients in σ/σ_0 are energy independent, and in this sense the new couplings are at an appropriate scale at every energy.

To complete this section we show that a transformation with the properties we require always exists, i.e. that we can find a matrix A that diagonalises $V(g)^{-1}$ in (22) and transforms $\hat{\sigma}_2$ in (23) to the unit matrix. The argument is analogous if one replaces $V(g)^{-1}$ with $V(\mathcal{O})$. By construction both $V(g)^{-1}$ and $\hat{\sigma}_2$ are symmetric and positive definite, so our problem is the same as finding normal coordinates for a multidimensional harmonic oscillator in classical mechanics (cf. e.g. [11]). To make this analogy transparent let us write $\mathcal{T} = \hat{\sigma}_2$ and $\mathcal{V} = V(g)^{-1}$; we then have to find A so that

$$A^T \cdot \mathcal{T} \cdot A = 1 \quad , \qquad A^T \cdot \mathcal{V} \cdot A = D \tag{29}$$

with D being diagonal. The elements d_i of D are generalised eigenvalues of $\mathcal{V}$ satisfying

$$\mathcal{V}\,\mathbf{a}_i = d_i\,\mathcal{T}\mathbf{a}_i \quad , \tag{30}$$

where $\mathbf{a}_i$ is the i-th column vector of A. The solution is well known to be

$$A = \mathcal{T}^{-1/2} \cdot U \tag{31}$$

where U is the orthogonal matrix that transforms $\mathcal{T}^{-1/2} \cdot \mathcal{V} \cdot \mathcal{T}^{-1/2}$ to D. Of course one need not use (31) in practice as there are convenient algorithms available to find A and D. In our numerical calculations we have used the routine `Eigenvals` of the algebraic package MAPLE.

4 Numerical examples

We will now give some numerical examples of our method described in the previous section. In this section we will stay within the framework of a leading order analysis of the observables. We start from the results in [9], where the sensitivity of optimal observables for semileptonic WW-decays was calculated. We assume a full kinematical reconstruction of the final state, except for the ambiguity one is left with if the jet charge is not known. For the standard model cross section we use the Born approximation and neglect effects of the finite W width. To describe the triple boson couplings we take the form factors f_i^γ, f_i^Z $(i = 1 \ldots 7)$ of [1]; deviations from their standard model tree level values will be referred to as "anomalous couplings".

Let us first look at a c.m. energy of $\sqrt{s} = 190$ GeV, which will be attained at LEP2. The coefficient matrix c can be found in Table 4 of [9] and in Table 1 here we give the diagonal elements of the transformed matrix c', ordered according to the symmetry of the corresponding observables. In appendix B we display the matrix A^{-1} for the transformation (20) between the coupling parameters. The one standard deviation ellipsoid is diagonal in the couplings g_i', thus its intersections with the coordinate axes equal its projections on these axes. The errors $\delta g_i'$ setting all other g_j' to zero are then equal to the errors $\Delta g_i'$ where all other g_j' are arbitrary. They are given by

$$\Delta g_i' = \frac{1}{\sqrt{V(g')^{-1}{}_{ii}}} = \frac{1}{\sqrt{N c_{ii}'}} \tag{32}$$

and are listed in Table 2 for an integrated luminosity of 500 pb^{-1}.

We immediately remark that a negative diagonal element occurs in the transformed coefficient matrix, which is not allowed because $c' = V(\mathcal{O}')$ is a covariance matrix and thus positive definite. We encounter here a problem of numerical instability: small errors in the calculation of the original matrices c and $\hat{\sigma}_2$ can have a large effect on the smallest generalised eigenvalues c_{ii}' and their eigenvectors, even to the point that eigenvalues come out with the wrong sign. This is not only a problem of our particular way of diagonalisation, but also occurs if one diagonalises c with an orthogonal matrix; we find that one of the usual eigenvalues of c in the subspace of couplings with symmetry (b) is negative. Such instabilities can cause large errors in the matrix inversion of c and $V(\mathcal{O})$. One needs $V(\mathcal{O})^{-1}$ to calculate the error on the extracted couplings as can be seen from (8) and (12), and large errors on c^{-1} can lead to large uncertainties in the extracted couplings, irrespective of whether c^{-1} is explicitly used to solve the system (7). One will of course aim to calculate c and $V(\mathcal{O})$ with best possible precision, but such an effort has limits, in particular if they are determined from simulated events and include for instance radiative corrections or detector effects. On a more fundamental level any calculation of these matrices will only be an approximation of the "exact" ones that correspond to the kinematical distributions seen in experiment. In this sense it seems quite inevitable that small eigenvalues (the usual or our generalised ones) of c and $V(\mathcal{O})$ and their eigenvectors are sensitive to imprecisions in the calculation and can lead to large errors or uncertainties in the data analysis. This holds of course even if one does not obtain eigenvalues with the wrong sign. We think that also in view of this a diagonalisation is useful, not because it solves the problem but because it makes it explicit! It allows to easily identify those combinations of

Table 1. Diagonal elements c_{ii}' of the transformed coefficient matrix at $\sqrt{s} = 190$ GeV. They are ordered by the symmetry classes (a) to (d) introduced in Sect. 2.1; the first row contains $c_{1,1}'$ to $c_{8,8}'$, the second $c_{9,9}'$ to $c_{16,16}'$ etc. Note that the c_{ii}' are by construction positive; the negative value for $c_{16,16}'$ is due to numerical integration errors in the original matrix c. This problem is further discussed in the text. A similar comment applies to Tables 3, 4 and 6

(a)	1.5	1.4	0.83	0.70	0.32	0.10	0.027	0.019
(b)	1.1	1.0	0.81	0.68	0.028	0.0093	0.0056	−0.00045
(c)	0.80	0.67	0.28	0.028	0.013	0.012		
(d)	1.4	1.2	0.85	0.13	0.039	0.017		

Table 2. 1–σ errors $\Delta g_i'$ on the extracted couplings corresponding to the coefficients c_{ii}' in Table 1. They are calculated from (32) for an integrated luminosity of 500 pb^{-1}

(a)	0.015	0.016	0.020	0.022	0.032	0.058	0.11	0.13
(b)	0.017	0.018	0.020	0.022	0.11	0.19	0.24	—
(c)	0.020	0.022	0.035	0.11	0.16	0.17		
(d)	0.015	0.017	0.020	0.051	0.093	0.14		

Table 3. Diagonal elements c_{ii}' of the coefficient matrix restricted to the left or right handed subspace of the couplings as explained in the text. The values in the left handed subspace differ from the corresponding ones in Table 1 by at most 3%

	left handed				right handed			
(a)	1.5	1.4	0.83	0.69	0.35	0.12	0.030	0.019
(b)	1.1	1.0	0.81	0.68	0.030	0.23	0.11	−0.00042
(c)	0.80	0.67	0.28		0.030	0.018	0.012	
(d)	1.4	1.2	0.85		0.13	0.042	0.018	

couplings which have small corresponding eigenvalues in c and $V(\mathcal{O})$ and will be the most unsafe ones in the analysis. From (32) we see that they are those combinations for which the statistical errors will be largest. Here the most unsafe coupling parameter is g'_{16}. One might thus choose to exclude it, and possibly other couplings, from the analysis and work in the remaining subspace of the g'_i where the numerics is more stable and where in any case the experiment is most sensitive. We will come back to this in Sect. 5.

In Tables 1 and 2 we find that the range of sensitivities is quite large, typically spanning several orders of magnitude in the c'_{ii}. We can actually identify the role of those form factors whose c'_{ii} are small. To this end we pass from the usual form factors f_i^γ, f_i^Z to the combinations which appear in the amplitudes for $e^+e^- \to W^+W^-$ with left or right handed electron polarisation, respectively:

$$\begin{aligned} f_i^L &= 4\sin^2\theta_W\, f_i^\gamma + (2 - 4\sin^2\theta_W)\,\xi f_i^Z, \\ f_i^R &= 4\sin^2\theta_W\, f_i^\gamma - 4\sin^2\theta_W\,\xi f_i^Z, \end{aligned} \tag{33}$$

where $\xi = s/(s - M_Z^2)$ and θ_W is the weak mixing angle. The matrix $\hat{\sigma}_2$ is block diagonal in the left and right handed form factors because different electron polarisations do not interfere in the cross section, but the coefficient matrix c is not. If one sets all right (left) handed anomalous couplings to zero then one has to diagonalise c in the left (right) handed subspace. Doing this we obtain the results shown in Table 3. In each of the classes (a) to (d) we find a clear correspondence of the generalised eigenvalues in the left (right) handed subspace with the largest (smallest) ones of the full matrix c given in Table 1. The form factors g'_i to which one is most sensitive thus correspond predominantly to left handed combinations, whereas the right handed combinations are more difficult to measure. This confirms our findings in [9] and can be explained by the missing neutrino exchange graph for right handed electrons which can give a large interference with anomalous triple boson couplings. As a word of caution we remark that the values given in Table 3 do not correspond to those for left or right handed electron beams, because for unpolarised beams both electron helicities contribute to the standard model cross section even if anomalous couplings corresponding to one helicity are (assumed to be) zero. We found however in [9] that the difference between c for a left handed electron beam and the left handed submatrix of c for unpolarised beams is small, again because the right handed standard model contribution is small compared to the left handed one due to the missing neutrino exchange graph.

Let us now come to the integrated cross section. We first give it in the parametrisation by f_i^γ and f_i^Z, where the standard model tree values are $f_1^\gamma = f_1^Z = 1$, $f_3^\gamma = f_3^Z = 2$ and zero for all other form factors:

$$\begin{aligned} \sigma/\sigma_0 = 1 &\\ &+ 0.022\,(\mathrm{Re} f_1^\gamma - 1) + 0.013\,(\mathrm{Re} f_1^Z - 1) - 0.031\,\mathrm{Re} f_2^\gamma \\ &- 0.010\,\mathrm{Re} f_2^Z - 0.074\,(\mathrm{Re} f_3^\gamma - 2) - 0.019\,(\mathrm{Re} f_3^Z - 2) \\ &+ 0.0058\,\mathrm{Re} f_5^\gamma \\ &+ 0.0092\,\mathrm{Re} f_5^Z + \{\text{quadratic terms}\} \end{aligned} \tag{34}$$

We do not give the full matrix $\hat{\sigma}_2$ for the quadratic terms here, but remark that the absolute values of its elements are between 0.001 and 0.3 and its eigenvalues between 0.004 and 0.4. In our new parametrisation we have

$$\begin{aligned} \sigma/\sigma_0 &= 1 + 0.18\,g'_1 + 0.16\,g'_2 - 0.0052\,g'_3 - 0.052\,g'_4 \\ &\quad - 0.15\,g'_5 + 0.071\,g'_6 - 0.029\,g'_7 + 0.0091\,g'_8 \\ &\quad + \sum_{i=1}^{28} (g'_i)^2 \\ &= 1 - 0.023 \\ &\quad + (g'_1 + 0.092)^2 + (g'_2 + 0.082)^2 + (g'_3 - 0.0026)^2 \\ &\quad + (g'_4 - 0.026)^2 + (g'_5 - 0.073)^2 + (g'_6 + 0.035)^2 \\ &\quad + (g'_7 - 0.014)^2 + (g'_8 + 0.0046)^2 + \sum_{i=9}^{28} (g'_i)^2 \end{aligned} \tag{35}$$

Comparing with the first row of Table 1 we see that couplings whose linear contribution to the cross section is relatively small can give a relatively large linear contribution to their optimal observable and vice versa. A measurement with an integrated luminosity of 500 pb^{-1} at $\sqrt{s} = 190$ GeV that finds the cross section equal to its standard model (Born level) value σ_0 will have a relative statistical error $\Delta\sigma/\sigma_{exp} = 1/\sqrt{N} = 0.018$. According to (27) this measurement would constrain the couplings g'_i to be between hyperspheres with radii $r_< = 0.066$ and $r_> = 0.202$ in their 28-dimensional space. Comparing with Table 2 we see that the thickness $r_> - r_< = 0.136$ of this shell is of the same order of magnitude as the largest statistical errors to be achieved with our optimal observables, which give the extensions of the 28-dimensional error ellipsoid in a linearised analysis. This reflects the well-known fact that the integrated cross section is clearly not as sensitive to anomalous couplings as the detailed kinematical distributions whose information we extract with our observables. It should however provide a useful cross check and help selecting the physical solution of (7) in a full nonlinear analysis. Moreover, even if the statistical errors using optimal observables are smaller in principle they might not be achievable in practice for couplings such as g'_{16} that are associated with large numerical instabilities, and in such cases the information from the total rate gives a valuable constraint.

The origin of the instabilities we encountered in the method of optimal observables is the large disparity of the eigenvalues of c, which we have seen to be associated with the large difference of their sensitivity to left and right handed combinations of form factors. In contrast the quadratic contribution to the integrated cross section is the same for left and right handed form factors f_i^L, f_i^R, so that the range of eigenvalues is smaller for $\hat{\sigma}_2$ than for c. As a consequence the numerical stability of $\hat{\sigma}_2$ is better than that of c, which we have checked by performing our diagonalisation (22), (23) with a fixed transformation matrix A for slightly different initial matrices c and $\hat{\sigma}_2$.

Let us now apply our method to a typical LC energy $\sqrt{s} = 500$ GeV with an integrated luminosity of 10 fb^{-1}

(cf. Tables 4 to 6). Here we encounter the particularity that the coefficients of the form factors $f_2^{\gamma,Z}$ and $f_7^{\gamma,Z}$ in the scattering amplitude grow faster with energy than those of the other couplings [1]. As a result the off-diagonal matrix elements in c between $f_2^{\gamma,Z}$ and another coupling are about a factor of 10 larger than matrix elements not involving $f_2^{\gamma,Z}$, and elements involving only $f_2^{\gamma,Z}$ are larger by yet another factor of 10. The (usual) eigenvalues of c in the *CP* conserving sector span 6 orders of magnitude. In the *CP* violating sector the situation is less dramatic, but still elements of c involving $f_7^{\gamma,Z}$ alone are about a factor of 10 larger than the others. The same phenomenon is found in the cross section, which reads

$$\begin{aligned}\sigma/\sigma_0 &= 1 \\ &\quad + 0.45\,(\mathrm{Re} f_1^\gamma - 1) + 0.23\,(\mathrm{Re} f_1^Z - 1) - 8.3\,\mathrm{Re} f_2^\gamma \\ &\quad - 4.3\,\mathrm{Re} f_2^Z - 0.58\,(\mathrm{Re} f_3^\gamma - 2) - 0.31\,(\mathrm{Re} f_3^Z - 2) \\ &\quad + 0.056\,\mathrm{Re} f_5^\gamma + 0.070\,\mathrm{Re} f_5^Z \\ &\quad + \{\text{quadratic terms}\} \end{aligned} \tag{36}$$

where $\hat{\sigma}_2$ has elements with absolute values between 0.04 and 1700 and eigenvalues between 0.02 and 1800. After our simultaneous diagonalisation the range of the matrix elements c'_{ii} is significantly smaller as can be seen in Table 4. The cross section now reads

$$\begin{aligned}\sigma/\sigma_0 &= 1 - 0.31\,g'_1 + 0.057\,g'_2 + 0.0018\,g'_3 - 0.014\,g'_4 \\ &\quad - 0.026\,g'_5 - 0.044\,g'_6 - 0.014\,g'_7 - 0.013\,g'_8 \\ &\quad + \sum_{i=1}^{28} (g'_i)^2 \\ &= 1 - 0.026 \\ &\quad + (g'_1 - 0.16)^2 + (g'_2 + 0.028)^2 + (g'_3 + 0.00091)^2 \\ &\quad + (g'_4 - 0.0068)^2 + (g'_5 - 0.013)^2 + (g'_6 - 0.022)^2 \\ &\quad + (g'_7 - 0.0069)^2 + (g'_8 - 0.0066)^2 + \sum_{i=9}^{28} (g'_i)^2 \quad . \end{aligned} \tag{37}$$

Of course, the g'_i are now in general energy-dependent for constant form factors $f_i^{\gamma,Z}$. As we mentioned in Sect. 3.1 the rapid growth with s of the coefficients of anomalous parts of the couplings $f_i^{\gamma,Z}$ in the amplitude has been "absorbed" into the coupling parameters by the transformation of $\hat{\sigma}_2$ to the unit matrix. As a result the linear coefficients in the cross section and the elements of the coefficient matrix c at $\sqrt{s} = 500$ GeV have the same order of magnitude as at LEP2 energies, and the large differences in the s-dependence of the coefficients for different couplings have been evened out.

A measurement at the LC that gives the standard model cross section constrains the couplings to be between 28-dimensional hyperspheres with radii $r_< = 0.140$ and $r_> = 0.182$, taking the relative statistical error of 0.0067 on the cross section one would obtain with an integrated luminosity of 10 fb^{-1}. The thickness of the shell, $r_> - r_< = 0.042$, is again of the order of the largest statistical errors on the g'_i one can obtain with optimal observables (cf. Table 5). Moreover such a small statistical error is likely to be small compared with systematic errors, so that the sensitivity of the integrated cross section will even be less.

Finally we remark that like in the case for $\sqrt{s} = 190$ GeV those couplings g'_i which give the largest statistical errors in the optimal observable analysis are predominantly related to right handed combinations of form factors as can be seen from the comparison of Tables 4 and 6.

Table 4. As Table 1, but for $\sqrt{s} = 500$ GeV

(*a*)	1.4	1.2	0.74	0.65	0.33	0.11	0.056	0.033
(*b*)	1.3	1.0	0.79	0.29	0.097	0.056	0.0092	−0.0013
(*c*)	1.2	0.58	0.32	0.066	0.027	0.013		
(*d*)	1.4	1.0	0.83	0.22	0.033	0.025		

Table 5. 1–σ errors $\Delta g'_i$ on the extracted couplings, corresponding to the coefficients c'_{ii} in Table 4 for an integrated luminosity of 10 fb^{-1}

(*a*)	0.0056	0.0062	0.0078	0.0083	0.012	0.020	0.029	0.037
(*b*)	0.0060	0.0066	0.0075	0.013	0.022	0.028	0.070	—
(*c*)	0.0062	0.0088	0.012	0.026	0.041	0.059		
(*d*)	0.0057	0.0067	0.0074	0.014	0.037	0.043		

Table 6. As Table 3 but for $\sqrt{s} = 500$ GeV, to be compared with Table 4

	left handed				right handed			
(*a*)	1.4	1.1	0.72	0.63	0.50	0.17	0.062	0.044
(*b*)	1.3	1.0	0.79	0.26	0.11	0.083	0.023	−0.0012
(*c*)	1.2	0.58	0.32		0.076	0.031	0.013	
(*d*)	1.4	1.0	0.83		0.24	0.040	0.026	

5 Simultaneous diagonalisation in practice

Let us sketch how our method of simultaneous diagonalisation might be used in practice.

1. One first has to choose which matrix to diagonalise simultaneously with $\hat{\sigma}_2$. These matrices need not be the same ones to be used in the data analysis itself but may be calculated under further approximations. Covariance matrices for the observables and extracted couplings can be evaluated for zero g_i as our entire procedure will only have its desired properties if nonlinear effects are not too large. If one uses the same approximation of the differential cross section (1) for the construction of the optimal observables (13) and the calculation of c, $V(\mathcal{O})$ and $N^{-1}V(g)^{-1}$ then the latter are all equal and can be diagonalised at the same time. Otherwise one has to choose a positive definite symmetric matrix for the diagonalisation, i.e. one of

the covariance matrices. The calculation of $V(g)$ or of its inverse from (12) involves however a matrix inversion and might suffer from numerical instabilities, so presumably the best choice will be $V(\mathcal{O})$.

2. In the next step one carries out the simultaneous diagonalisation of the chosen matrix and $\hat{\sigma}_2$ as described in Sect. 3.1 and determines the transformation matrix A in (20) to (22). At this point it will be useful to test the numerical stability of the transformed matrices, for instance by re-calculating them in the new basis of couplings or by repeating the diagonalisation procedure with slightly modified initial matrices. One might choose to discard some of the new couplings g_i' and the corresponding observables from the analysis if the corresponding matrix elements are found to be instable. This does not mean that one has to set these couplings to zero. From the measurement of the total cross section one will obtain limits on them, which will also allow to control the contribution they can give to the mean values of those observables that are kept in the analysis because the matrix c' is not exactly diagonal and because of nonlinear terms in (7).
3. In the new parametrisation of the couplings one then carries out the analysis of the data. Here $V(\mathcal{O}')$, c', $\hat{\sigma}_2'$ and the other coefficients in (4) will be determined under the most realistic assumptions and with the best precision one can afford. They will not be exactly diagonal in practice, but should have small off-diagonal elements if the approximations made in step 1. and in the construction of the optimal observables are sufficiently good.
4. One can then give both single and multidimensional errors on the measured coupling parameters g_i'. At this final stage results can also be presented in more conventional and process independent parametrisations of the couplings. Couplings g_i' with large errors will then correspond to badly constrained directions in parameter space. Using the function χ^2 of (8) one can also obtain couplings and their errors in various subspaces as explained in Sect. 2. Such subspaces may be motivated by physical considerations or correspond to restricted parameter sets used by other experiments one wants to compare with.

6 Summary

In the first part of this paper we have discussed how to extract coupling parameters from the measured mean values $\overline{\mathcal{O}}_i$ of appropriate observables without the approximation that the couplings are small. A study of triple gauge couplings with this method using generated events has been performed in [14]. Errors on the couplings can be obtained from a least squares fit of the $\overline{\mathcal{O}}_i$. If one puts constraints on the couplings in order to reduce their number the method also gives an indication of how compatible these constraints are with the data. The "optimal observables" discussed in [9] have statistical errors equal to the smallest possible ones to leading order in the coupling parameters g_i. Beyond the leading order approximation one can obtain more sensitive observables if one has some previous estimate $\tilde{g}_i$ for the couplings, expanding the differential cross section around $\tilde{g}_i$ instead of zero and constructing observables from the corresponding expansion coefficients. In appendix A we show that up to linear reparametrisations the choice of optimal observables is unique: any other set of observables must give bigger (statistical) errors. A consequence of this is that *no* set of observables gives optimal errors for all actual values of the coupling parameters.

In a second part we have proposed to perform the data analysis using a particular parametrisation g_i' of the couplings, which is specific to the process and its c.m. energy. It is obtained from the initial set g_i by a linear transformation which diagonalises the covariance matrix $V(\mathcal{O})$ of the observables and transforms the matrix $\hat{\sigma}_2$ of quadratic coefficients in the integrated cross section (2) to unity. In an idealised framework each optimal observable $\mathcal{O}_i'$ for this parametrisation is only sensitive to one coupling, and the statistical errors on the extracted couplings are uncorrelated. Under realistic circumstances both properties can be approximately satisfied provided that the analysis stays in a region of parameter space where the dependence of the mean values $\overline{\mathcal{O}}_i'$ on the couplings is not far from linear. Various matrices are then approximately diagonal which should generally facilitate the data analysis. In particular one can directly give errors on single or a small number of couplings, which will be necessary to obtain statistically significant results with a limited number of events. At the same time one can readily present multidimensional errors in parameter space, which is essential to compare with the results of measurements that impose various different constraints on the couplings. Having approximately diagonal matrices also allows to easily identify those directions in parameter space which can be measured best and those for which the statistical errors will be large and which are likely to be associated with numerical instabilities, for example in matrix inversions. One can thus recognise and seek to remedy such problems in an early stage of the analysis.

The measurement of the total cross section σ gives valuable complementary information on the coupling parameters. Its dependence on the couplings is particularly simple in the parametrisation we propose since the quadratic contributions are $(g_i')^2$ times the standard model cross section σ_0, i.e. they have the same form for all couplings. A measurement of σ will then restrict the g_i' to a shell between two hyperspheres in parameter space.

We have given some numerical examples of our method applied to the semileptonic decay channels in $e^+e^- \to W^+W^-$. In particular we find that the couplings g_i' which can be measured best with unpolarised beams predominantly appear in the amplitude for left handed electrons (or right handed positrons), and that the g_i' with the largest statistical errors mainly correspond to the opposite lepton helicity. Comparing our results at LEP2 and LC energies we see that the coefficients in the linear contributions of the couplings g_i' to our observables and to the integrated cross section change much less with en-

ergy than in usual parametrisations. This is because in the new parametrisation the quadratic coefficients in the normalised cross section σ/σ_0 are by construction energy independent.

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Appendix A: Uniqueness of optimal observables

In this appendix we show that the set of observables (16), obtained from expanding the differential cross section about the actual values of the couplings, is unique in the sense that up to the linear reparametrisations (39) it is the only set of n integrated observables which in the limit of large N leads to the minimum error on the n extracted parameters.

To keep our notation simple we give the proof for the case that the actual values of the g_i are zero. The expectation value and covariance of functions $f(\phi)$ and $g(\phi)$ are then given by

$$E_0[f] = \frac{\int d\phi\, f(\phi) S_0(\phi)}{\int d\phi\, S_0(\phi)} \quad ,$$
$$V_0[f,g] = E_0[fg] - E_0[f]\, E_0[g] \quad . \tag{38}$$

In the general case one has instead of S_0 the zeroth order coefficient $\widetilde{S}_0$ (14) from the expansion about the appropriate values $\tilde{g}_i$.

For large N the covariance matrix for the extracted couplings is given by (12). Under a linear reparametrisation of observables,

$$\mathcal{O}_i(\phi) \to \mathcal{O}'_i(\phi) = \sum_j a_{ij} \mathcal{O}_j(\phi) + b_i \quad , \tag{39}$$

where a_{ij} and b_i are constants and the matrix a_{ij} is nonsingular, the matrices c from (6) and $V(\mathcal{O})$ transform according to

$$c' = a \cdot c \quad ,$$
$$V(\mathcal{O}') = a \cdot V(\mathcal{O}) \cdot a^T \quad . \tag{40}$$

From (12) we see that the covariance matrix $V(g)$ is unchanged under such a transformation. For our proof we can hence restrict ourselves to observables with mean value

$$E_0[\mathcal{O}_i] = 0 \tag{41}$$

and with a coefficient matrix $c_{ij} = \delta_{ij}$. From (6) we then have the condition

$$V_0[\mathcal{O}_i\,,\ S_{1,j}/S_0] = \delta_{ij} \tag{42}$$

and the error on the extracted couplings is given by

$$V(g)_{ij} = N^{-1} V(\mathcal{O})_{ij} = N^{-1} V_0[\mathcal{O}_i, \mathcal{O}_j] \quad . \tag{43}$$

From [9] we know that the optimal observables (13) lead to the smallest possible error on the g_i, given by the

Table 7. Blocks of the matrix A^{-1} for $\sqrt{s} = 190$ GeV, corresponding to symmetry classes (a) to (d). All matrix elements are multiplied by a factor of 100

(a)	$\mathrm{Re}\bar{f}_1^\gamma$	$\mathrm{Re}\bar{f}_1^Z$	$\mathrm{Re}f_2^\gamma$	$\mathrm{Re}f_2^Z$	$\mathrm{Re}\bar{f}_3^\gamma$	$\mathrm{Re}\bar{f}_3^Z$	$\mathrm{Re}f_5^\gamma$	$\mathrm{Re}f_5^Z$
g'_1	4.7	6.0	−7.2	−9.3	−19	−23	−1.4	−4.8
g'_2	3.0	3.5	−4.1	−4.9	−7.4	−11	5.9	9.6
g'_3	2.2	2.1	−7.4	−10	−1.8	−0.66	−0.59	−0.57
g'_4	−11	−15	6.4	7.0	5.8	3.2	0.93	0.12
g'_5	−2.1	7.0	4.6	−11	13	−26	−2.8	2.4
g'_6	2.1	−5.6	−4.9	7.8	−7.8	8.4	−5.9	6.6
g'_7	7.8	−11	−7.5	11	−1.8	3.9	1.7	−2.8
g'_8	7.0	−9.3	−0.29	0.57	−4.2	5.7	−0.94	1.3

(b)	$\mathrm{Im}\bar{f}_1^\gamma$	$\mathrm{Im}\bar{f}_1^Z$	$\mathrm{Im}f_2^\gamma$	$\mathrm{Im}f_2^Z$	$\mathrm{Im}\bar{f}_3^\gamma$	$\mathrm{Im}\bar{f}_3^Z$	$\mathrm{Im}f_5^\gamma$	$\mathrm{Im}f_5^Z$
g'_9	−0.67	−1.4	−3.5	−5.6	−14	−22	−1.1	−3.1
g'_{10}	0.69	1.4	−2.2	−2.7	−4.5	−7.2	6.1	10
g'_{11}	−8.8	−14	3.0	5.4	9.2	14	−0.34	1.0
g'_{12}	−7.3	−12	10	16	7.3	12	−0.0021	1.6
g'_{13}	11	−13	−8.4	9.8	−6.0	6.7	−0.70	0.83
g'_{14}	1.6	−3.3	4.6	−5.2	0.27	1.6	1.7	−1.0
g'_{15}	0.72	2.5	0.074	−3.8	−2.8	−2.5	−6.4	7.8
g'_{16}	3.9	−5.3	−6.8	9.1	−18	23	0.45	−0.51

(c)	$\mathrm{Re}f_4^\gamma$	$\mathrm{Re}f_4^Z$	$\mathrm{Re}f_6^\gamma$	$\mathrm{Re}f_6^Z$	$\mathrm{Re}f_7^\gamma$	$\mathrm{Re}f_7^Z$
g'_{17}	8.3	14	6.1	13	−9.7	−14
g'_{18}	−7.7	−12	−2.8	−10	−15	−27
g'_{19}	4.4	5.6	−24	−41	−11	−19
g'_{20}	−11	14	−9.3	8.0	3.4	−5.6
g'_{21}	1.3	−3.8	−26	29	−18	20
g'_{22}	−5.5	7.3	6.2	−8.4	−13	17

(d)	$\mathrm{Im}f_4^\gamma$	$\mathrm{Im}f_4^Z$	$\mathrm{Im}f_6^\gamma$	$\mathrm{Im}f_6^Z$	$\mathrm{Im}f_7^\gamma$	$\mathrm{Im}f_7^Z$
g'_{23}	3.9	7.6	−24	−39	−11	−18
g'_{24}	−11	−18	−11	−13	−5.5	−5.9
g'_{25}	−0.38	0.22	−1.1	−2.7	19	28
g'_{26}	−8.4	12	−16	24	−2.6	3.9
g'_{27}	8.7	−10	−20	24	−7.4	10
g'_{28}	1.6	−3.2	4.6	−6.6	20	−27

Cramér-Rao bound. To satisfy our conditions (41) and (42) we take the linear combinations

$$\mathcal{D}_i(\phi) = \sum_j (c_{opt})^{-1}{}_{ij} \left(S_{1,j}(\phi)/S_0(\phi) - E_0[S_{1,j}/S_0] \right) \tag{44}$$

with

$$(c_{opt})_{ij} = V_0[S_{1,i}/S_0\,,\; S_{1,j}/S_0] \quad . \tag{45}$$

We assume that the functions $S_{1,i}$ are linearly independent. Otherwise some of the parameters g_i are superfluous and can be eliminated; our assumption is thus that the g_i are an independent set of parameters for the anomalous couplings. Linear independence of the $S_{1,i}(\phi)$ guarantees that c_{opt} is nonsingular, which has tacitly been used at several instances in our paper. The set $\mathcal{D}_i$ is related to the optimal observables $S_{1,i}/S_0$ by a linear transformation (39) and thus gives the same optimal error matrix $V(g)$.

The covariance $V_0[f,g]$ defines a scalar product on the Hilbert space $\mathcal{H}$ of sufficiently smooth functions of ϕ with the property $E_0[f] = 0$.[2] The functions $\mathcal{D}_i$ span a subspace $\mathcal{H}^I$ of $\mathcal{H}$, and we define $\mathcal{H}^{II}$ as the orthogonal complement of $\mathcal{H}^I$ with respect to the scalar product $V_0[f,g]$. Any set of n observables satisfying (41) can then be written as

$$\mathcal{O}_i = \mathcal{O}_i^I + \mathcal{O}_i^{II} \tag{46}$$

with $\mathcal{O}_i^I \in \mathcal{H}^I$, $\mathcal{O}_i^{II} \in \mathcal{H}^{II}$. Further decomposing $\mathcal{O}_i^I = \sum_j a_{ij}\mathcal{D}_j$ and using the constraint (42) we obtain $a_{ij} = \delta_{ij}$, i.e.

$$\mathcal{O}_i^I = \mathcal{D}_i \quad . \tag{47}$$

Finally, we have from (43), (46), (47)

$$V(g)_{ij} = N^{-1}V_0[\mathcal{D}_i, \mathcal{D}_j] + N^{-1}V_0[\mathcal{O}_i^{II}, \mathcal{O}_j^{II}] \quad . \tag{48}$$

The first term gives the error on the couplings for the optimal observables $\mathcal{D}_i$, which is minimal. If the observables $\mathcal{O}_i$ have minimal error, too, the second term must be zero, so that for each i we have $V_0[\mathcal{O}_i^{II}, \mathcal{O}_i^{II}] = 0$ and thus

$$\mathcal{O}_i^{II} = 0 \quad , \tag{49}$$

which completes our proof.

In Sect. 3.1 we mentioned that instead of $\overline{\mathcal{O}}_i$ one may use the product $N\overline{\mathcal{O}}_i$ measured with fixed luminosity to extract the couplings [9]. By an argument analogous to the one of this appendix one finds that up to linear reparametrisations our observables (16) are again the only optimal ones. In this case linear reparametrisations have to be homogeneous, i.e. one must have $b_i = 0$ in (39), since adding constants to the observables can change the induced errors on the coupling parameters.

Appendix B: Transformation matrices at $\sqrt{s} = 190$ GeV

In Table 7 we give the matrix A^{-1} for the transformation (20) to our g_i' at $\sqrt{s} = 190$ GeV, where the initial parameters g_i are the real and imaginary parts of the form factors f_i^γ, f_i^Z minus their standard model tree level values. For brevity we write $\bar{f}_1^\gamma = f_1^\gamma - 1$, $\bar{f}_1^Z = f_1^Z - 1$, $\bar{f}_3^\gamma = f_3^\gamma - 2$, $\bar{f}_3^Z = f_3^Z - 2$. The matrix A^{-1} is block diagonal in the symmetry classes (a) to (d) introduced in Sect. 2.1. For convenience the matrix elements are multiplied by a factor of 100 in the table.

From these matrices and the errors $\Delta g_i'$ in Table 2 on can read off to which combinations of the f_i^γ, f_i^Z the experiment is most and to which it is least sensitive. Corresponding matrices for other parametrisations of the couplings, e.g. those of [3], can easily be obtained by further transformation from the f_i^γ, f_i^Z to these parameter sets.

[2] A similar scalar product has also been used in [4]

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