Role of Fermi Statistics in Glauber Scattering Amplitudes

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Contents

- Role of Fermi-Statistics in Atomic and Nuclear Physics
  The interaction of two atoms at short distances when their electron clouds are overlapped contains additional terms due to Fermi statistics of electrons. If electrons were bosons the signs of the exchange integrals would be opposite to those for fermions.

- Two toy examples.
  For ground state of the Helium nucleus with S=I=0 and the totally antisymmetric spin-isospin wave function the space part of the nuclear wave function is totally symmetric. All nucleons are in S-wave, hence
  $|\Psi(\vec{r}_1,\vec{r}_2,\vec{r}_3,\vec{r}_4)|^2 = |\psi_s(\vec{r}_1)|^2|\psi_s(\vec{r}_2)|^2|\psi_s(\vec{r}_3)|^2|\psi_s(\vec{r}_4)|^2 = \rho(\vec{r}_1)\cdot\rho(\vec{r}_2)\cdot\rho(\vec{r}_3)\cdot\rho(\vec{r}_4).$

  For exited state of the Helium nucleus with S=I=1 and the totally symmetric spin-isospin wave function the space part of the nuclear wave function is totally antisymmetric. Therefore $|\Psi(\vec{r}_1,\vec{r}_2,\vec{r}_3,\vec{r}_4)|^2 = 0$ when $\vec{r}_j = \vec{r}_m$ for any j and m, which means that the approximation

$$|\Psi(\vec{r}_1,\vec{r}_2,\vec{r}_3,\vec{r}_4)|^2 = \rho(\vec{r}_1)\cdot\rho(\vec{r}_2)\cdot\rho(\vec{r}_3)\cdot\rho(\vec{r}_4)$$

  is totally wrong.

- Basic Formulas of Glauber Theory

Amplitude of hadron-nucleus scattering is

$$F_{fi}(\vec{q}) = \frac{ik}{2\pi}\int \exp\{i\vec{q}\vec{b}\}\langle\Psi_f|\Gamma(\vec{b},\vec{s}_1,\vec{s}_2,...\vec{s}_A)|\Psi_i\rangle,$$

where $|\Psi_i\rangle$ describes initial and $|\Psi_f\rangle$ final state of nucleus. Profile function of nucleus $\Gamma(\vec{b},\vec{s}_1,\vec{s}_2,...\vec{s}_A)$ is expressed in terms of A hadron-nucleon profile functions $\gamma_j \equiv \gamma(\vec{b}-\vec{s}_j), \quad j = 1,2,...,A$:

$$\Gamma(\vec{b},\vec{s}_1,\vec{s}_2,...\vec{s}_A) = 1-\prod_{j=1}^{A}[1-\gamma(\vec{b}-\vec{s}_j)] = \sum_{j=1}^{A}\gamma_j - \sum_{j<l\leq A}\gamma_j\gamma_l + \sum_{j<l<n\leq A}\gamma_j\gamma_l\gamma_n - ...,$$

$$\gamma(\vec{b}) = \frac{1}{2i\pi k}\int \exp\{-i\vec{q}\vec{b}\}f(\vec{q})d^2\vec{q}.$$

If beam momentum $\vec{k}$ is directed along Z-axis then $\vec{b}$ and $\vec{s}_j$ belong to XY-plane (two dimentional vectors).

Properties of Exact Formulas and Approximations

- Amplitude of hadron-nucleus scattering is expressed in terms of the nuclear wave functions of initial (i) and final (f) states and the hadron-nucleon (vacuum) amplitudes.
- Approximate formulas for elastic scattering $(i = f)$

$$|\Psi_i(\vec{r}_1, \vec{r}_2, \vec{r}_3, ..., \vec{r}_A)|^2 = \rho(\vec{r}_1) \cdot \rho(\vec{r}_2) \cdot \rho(\vec{r}_3) \cdot ... \cdot \rho(\vec{r}_A),$$

$$\langle \Psi_i | \Gamma(\vec{b}, \vec{s}_1, \vec{s}_2, ...\vec{s}_A) | \Psi_i \rangle = 1 - \prod_{j=1}^{A} [1 - \int \gamma(\vec{b} - \vec{s}_j) \rho(\vec{r}_j) d^3\vec{r}_j] =$$

$$= 1 - [1 - \int \gamma(\vec{b} - \vec{s}_1) \rho(\vec{r}_1) d^3\vec{r}_1]^A.$$

Approximate formulas permit to express many-dimensional integral through three-dimensional integral.

- Using of complete set of nuclear wave functions

$$\langle \Psi_i | \gamma(\vec{b} - \vec{s}_1) \gamma(\vec{b} - \vec{s}_2) | \Psi_i \rangle = \sum_n \langle \Psi_i | \gamma(\vec{b} - \vec{s}_1) | \Psi_n \rangle \langle \Psi_n | \gamma(\vec{b} - \vec{s}_2) | \Psi_i \rangle \approx$$

$$\approx \langle \Psi_i | \gamma(\vec{b} - \vec{s}_1) | \Psi_i \rangle \langle \Psi_i | \gamma(\vec{b} - \vec{s}_2) | \Psi_i \rangle = \int \gamma(\vec{b} - \vec{s}_1) \rho(\vec{r}_1) d^3\vec{r}_1 \int \gamma(\vec{b} - \vec{s}_2) \rho(\vec{r}_2) d^3\vec{r}_2.$$

- Refutation of the proof

  Let us consider the general ($f \neq i$) matrix element $\langle \Psi_f | \gamma(\vec{b} - \vec{s}_1)\gamma(\vec{b} - \vec{s}_2) | \Psi_i \rangle$
  Applying the permutation of particles 2 and 3 (23), then due to Fermi statistics

  $(23)|\Psi_i\rangle = -|\Psi_i\rangle$, $\langle \Psi_n|(23) = -\langle \Psi_n|$.
  Therefore $\langle \Psi_f|\gamma_1\gamma_2|\Psi_i\rangle = \sum_n \langle \Psi_f|\gamma_1|\Psi_n\rangle\langle \Psi_n|\gamma_2|\Psi_i\rangle =$
  $= \sum_n \langle \Psi_f|\gamma_1|\Psi_n\rangle\langle \Psi_n|(23)\gamma_2(23)|\Psi_i\rangle = \sum_n \langle \Psi_f|\gamma_1|\Psi_n\rangle\langle \Psi_n|\gamma_3|\Psi_i\rangle =$
  $= \langle \Psi_f|\gamma_1\gamma_3|\Psi_i\rangle$.
  Since $\langle \Psi_f|\gamma_1[\gamma_2 - \gamma_3]|\Psi_i\rangle \equiv 0$ for any f and i of the complete set of functions, then $\gamma(\vec{b} - \vec{s}_2) \equiv \gamma(\vec{b} - \vec{s}_3)$. But this is nonsense!

- What is wrong?

  In the space of totally antisymmetric wave functions the admissible operators B are those which transform antisymmetric functions into antisymmetric ones. For instance, $B_m = \sum_{j=1}^{A} [\gamma_j]^m$ obey this demand. But $\gamma_j|\Psi_i\rangle$ is not totally antisymmetric function, hence operators like $[\gamma_j]^m$ (without sum over all j, $1 \leq j \leq A$) cannot be considered for nucleons obeying Fermi-statistics.

Properties of Exact Formulas and Approximations

- True application of completeness condition

$$1 - \gamma(\vec{b} - \vec{s}_j) = \exp\{\ln[1 - \gamma_j]\} = \exp\{-\gamma_j - [\gamma_j]^2/2 - [\gamma_j]^3/3 - [\gamma_j]^4/4 - ...\},$$

if $|\gamma_j| < 1$. Hence $\prod_{j=1}^{A}[1 - \gamma_j] = \exp\{-\xi\}$, with

$$\xi = \sum_{j=1}^{A}\{\gamma_j + [\gamma_j]^2/2 + [\gamma_j]^3/3 + [\gamma_j]^4/4 - ...\} = \sum_{m=1}^{\infty}\sum_{j=1}^{A}\frac{[\gamma_j]^m}{m}.$$

Averaging over the wave function gives

$$\langle\Psi_i|\exp\{-\xi\}|\Psi_i\rangle = \langle\Psi_i|[1 - \frac{\xi}{1!} + \frac{\xi^2}{2!} - \frac{\xi^3}{3!} - ...]|\Psi_i\rangle =$$

$$= 1 - \langle\Psi_i|\xi|\Psi_i\rangle + \frac{1}{2!}\sum_n\langle\Psi_i|\xi|\Psi_n\rangle\langle\Psi_n|\xi|\Psi_i\rangle - ... \approx \exp\{-\langle\Psi_i|\xi|\Psi_i\rangle\}.$$

Finally for elastic scattering, we have

$$\langle\Psi_i|\Gamma(\vec{b}, \vec{s}_1, \vec{s}_2, ...\vec{s}_A)|\Psi_i\rangle \approx 1 - \exp\{-\langle\Psi_i|\xi|\Psi_i\rangle\}.$$

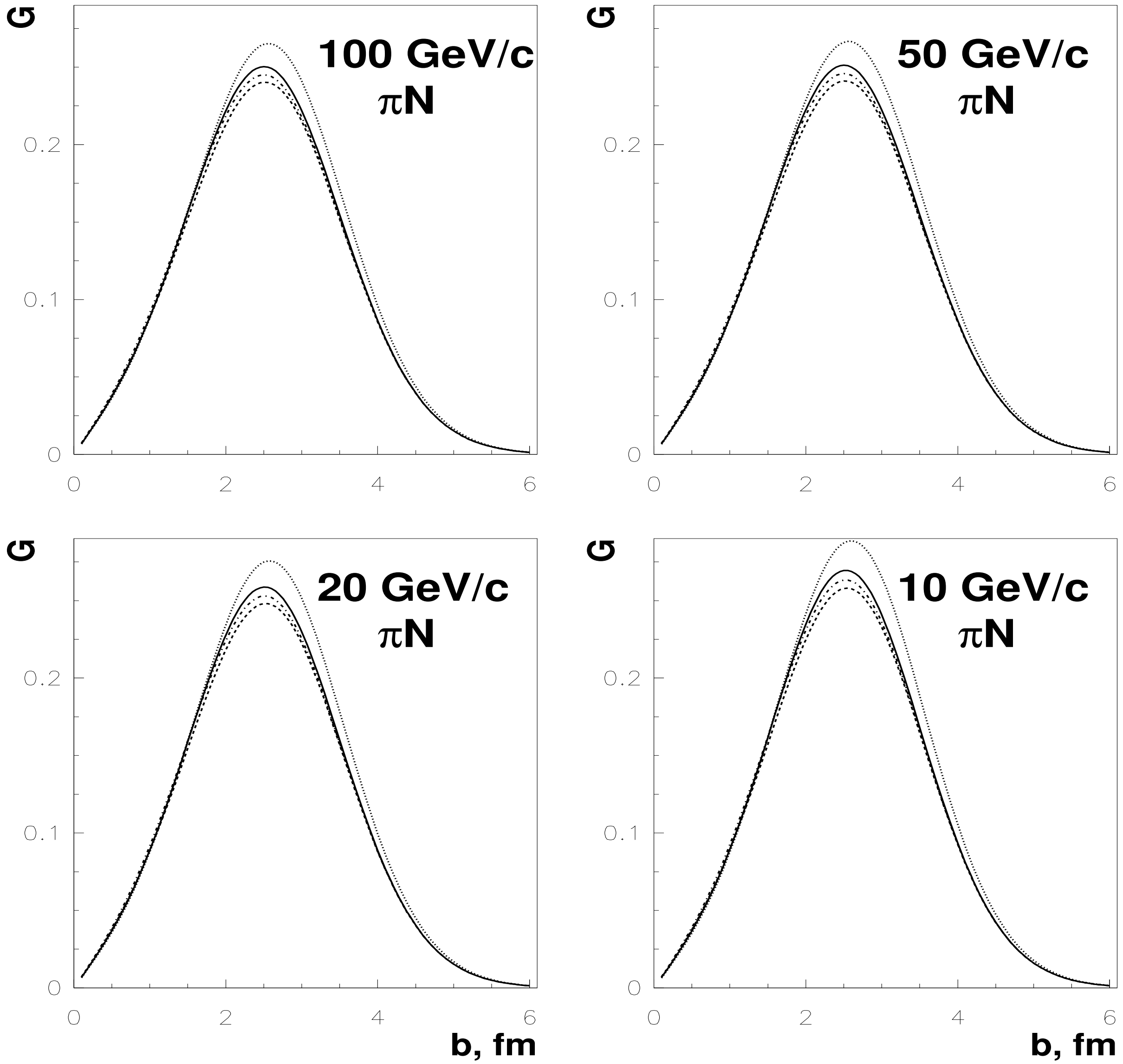
- Elastic scattering on ground state of Oxigen ($S = I = 0$, $A = 16$)
  S and P nucleons, harmonic oscillator wave functions.
  πN and pN amplitudes from J.P. Burq et al, Nucl. Phys. B217 (1983) 285.
- Amplitudes of $\pi^{16}O$ and $p^{16}O$ elastic scattering at $10 \leq k \leq 100$ GeV/c

$$F_{ii}(q) = \frac{ik}{q}\int_0^\infty J_1(qb)G(b)bdb$$

  where $J_n(z)$ denotes the Bessel function and
  $G(b) = -\frac{d}{db}\langle\Psi_i|\Gamma(\vec{b},\vec{s}_1,\vec{s}_2,...,,\vec{s}_A)|\Psi_i\rangle$.
  - Exact calculation with Slater determinant
    (R.H. Bassel, C. Wilkin, Phys. Rev. 174 (1968) 1179.)
  - Using completeness relation
    $\langle\Psi_i|\Big\{\Gamma(\vec{b},\vec{s}_1,\vec{s}_2,...,,\vec{s}_A)\Big\}|\Psi_i\rangle = 1 - \exp\{-\langle\Psi_i|\xi|\Psi_i\rangle\}$,
    where $\xi = \sum_{j=1}^{A}\{\gamma_j + [\gamma_j]^2/2 + [\gamma_j]^3/3 + [\gamma_j]^4/4 + [\gamma_j]^5/5 + ...\}$.
  - Approximation of independent particles
    $|\Psi_i(\vec{r}_1,\vec{r}_2,...,\vec{r}_A)|^2 = \rho(\vec{r}_1)\cdot\rho(\vec{r}_2)\cdot...\cdot\rho(\vec{r}_A)$,
    hence $\langle\Psi_i|\Big\{\Gamma(\vec{b},\vec{s}_1,\vec{s}_2,...,,\vec{s}_A)\Big\}|\Psi_i\rangle = 1 - [1-\eta]^A$
    with $\eta = \int\gamma(\vec{b}-\vec{s}_1)\rho(\vec{r}_1)d^3\vec{r}_1$.
  - Optical limit $\langle\Psi_i|\Big\{\Gamma(\vec{b},\vec{s}_1,\vec{s}_2,...,,\vec{s}_A)\Big\}|\Psi_i\rangle = 1 - \exp\{-A\eta\}$.

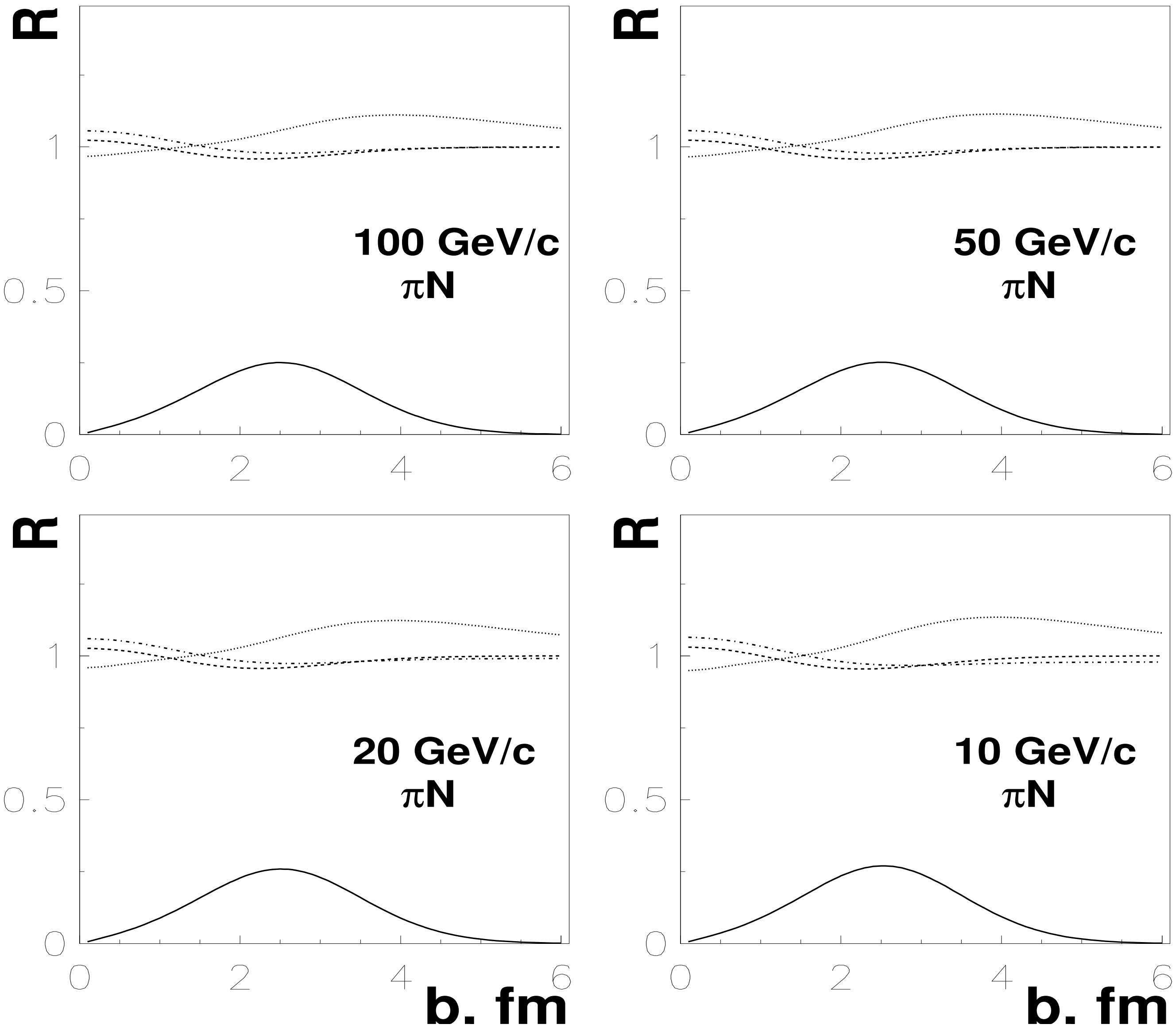
Results of Numerical Calculation of Profile Function

- Real Part of Profile Functin for π^{16}O scattering $G = -\frac{d}{db}\langle\Psi_i|\Gamma(\vec{b},\vec{s}_1,...,\vec{s}_A)|\Psi_i\rangle$



Results of Numerical Calculation of Profile Function

- Ratios R of Appriximate Profile Functins to Exact One for π^{16}O scattering



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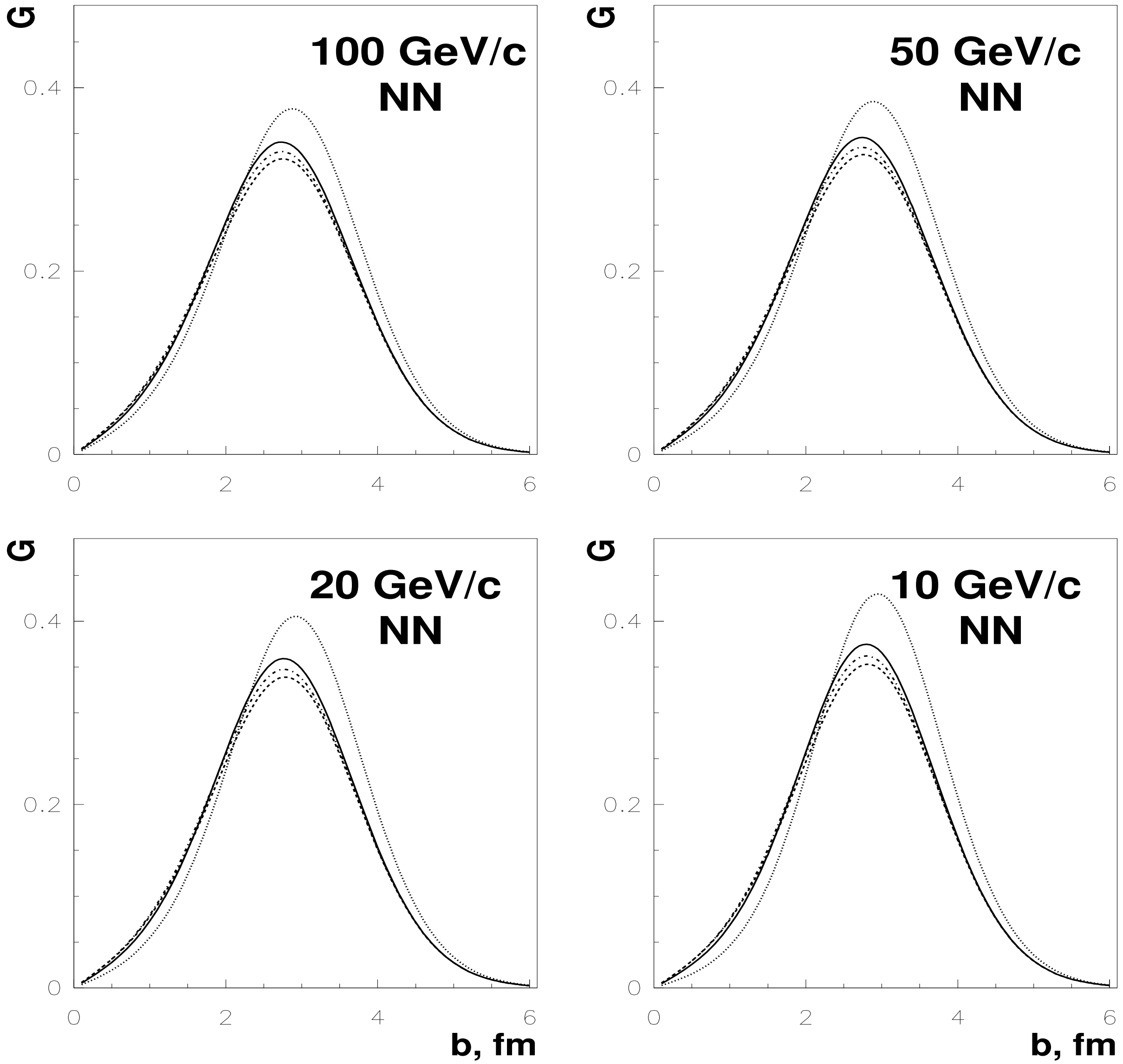

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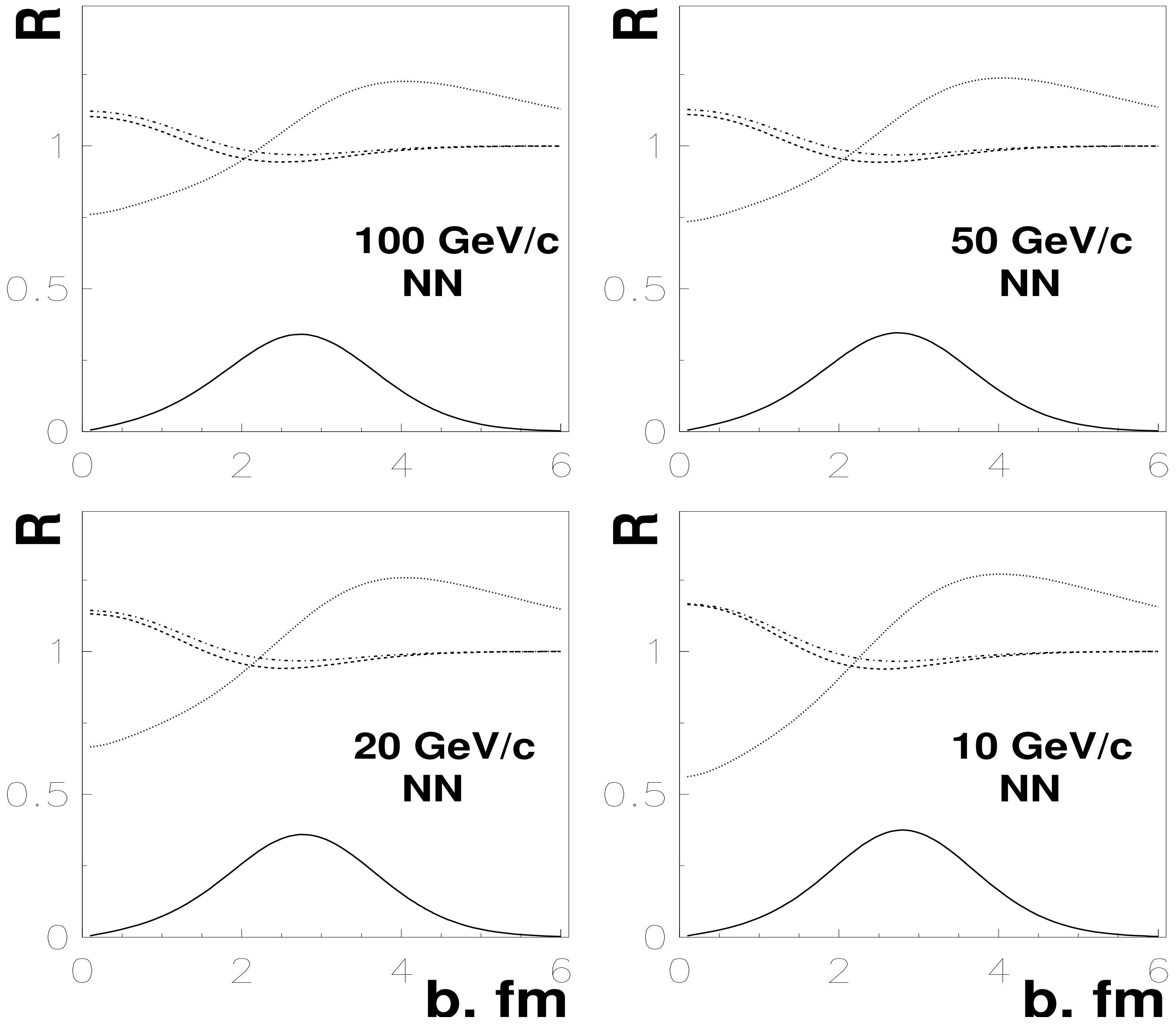
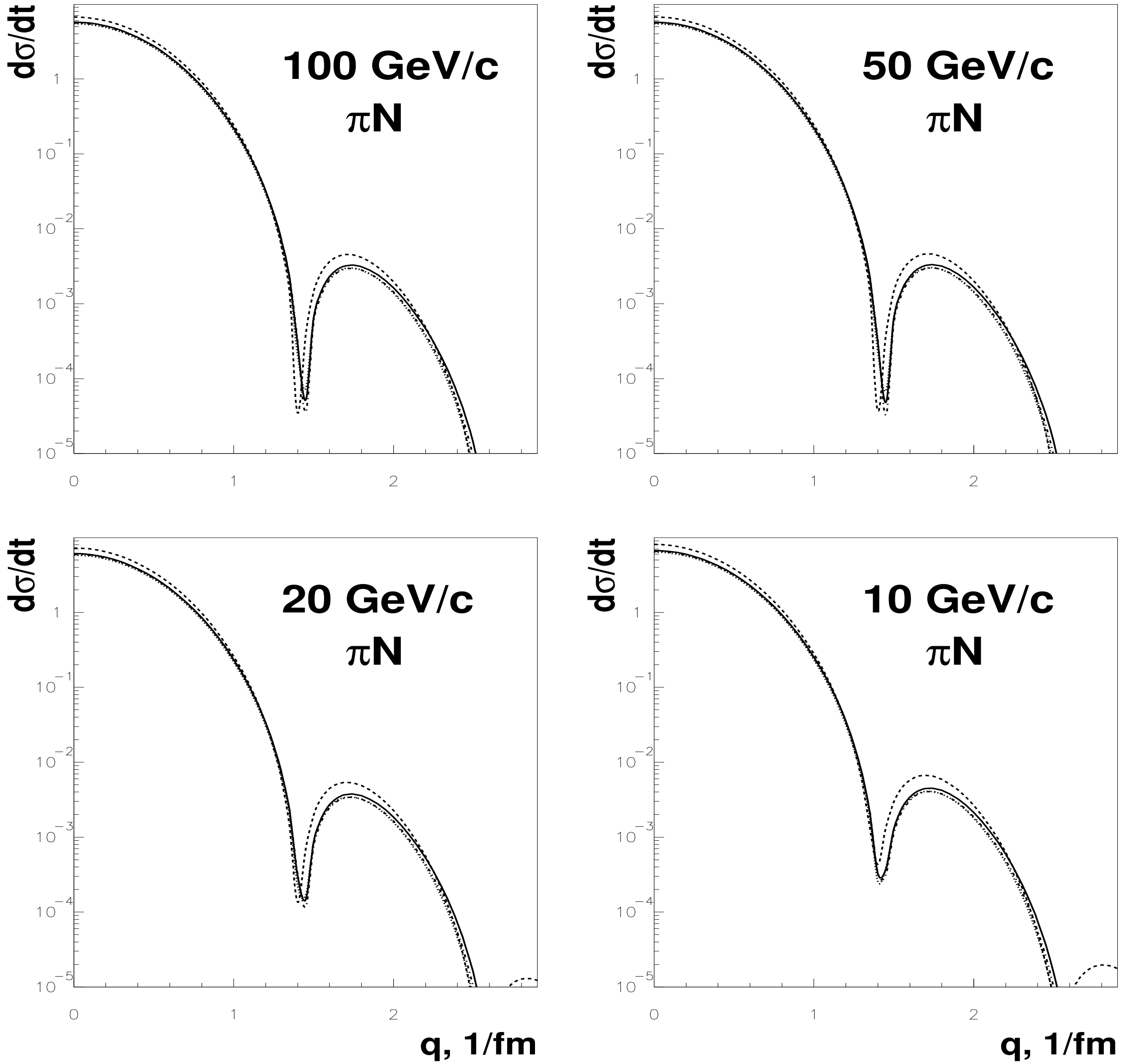


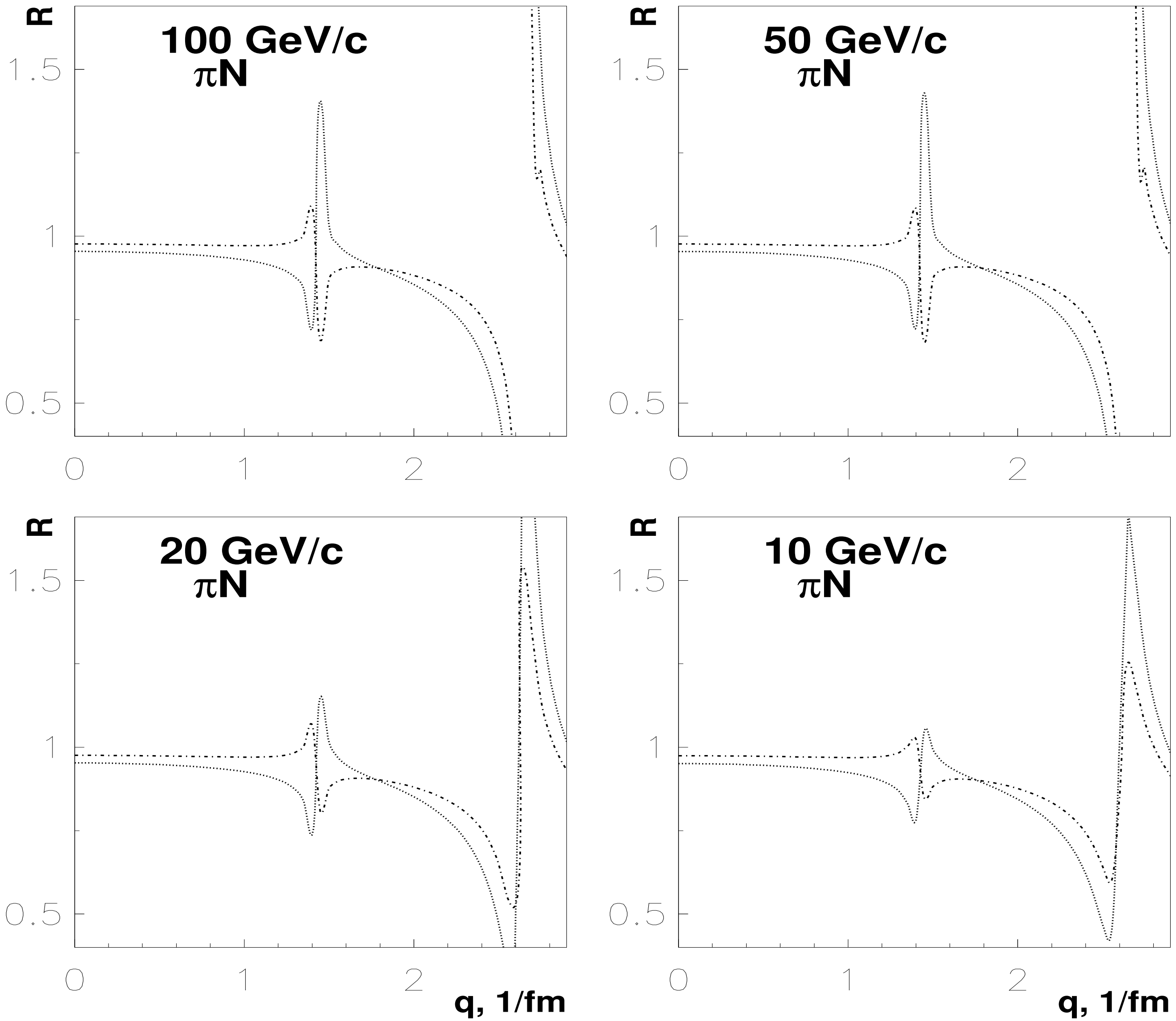
Results of Numerical Calculation of Differential Cross Section

- Differential Cross Section $d\sigma/dt$ for π^{16}O scattering



Results of Numerical Calculation of Differential Cross Section

- Ratios of Approximate Differential Cross Sections for π^{16}O Scattering to Exact One



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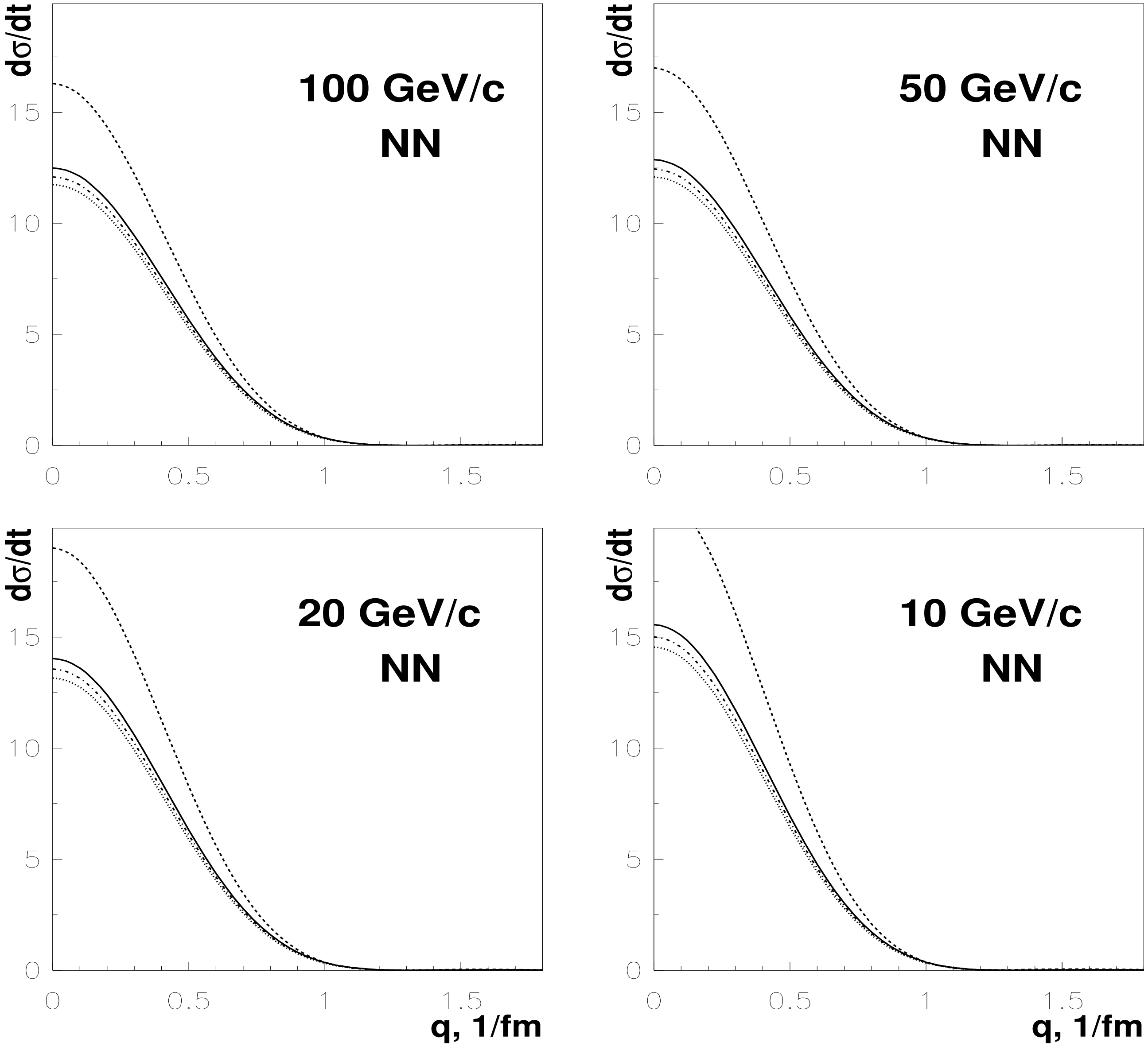

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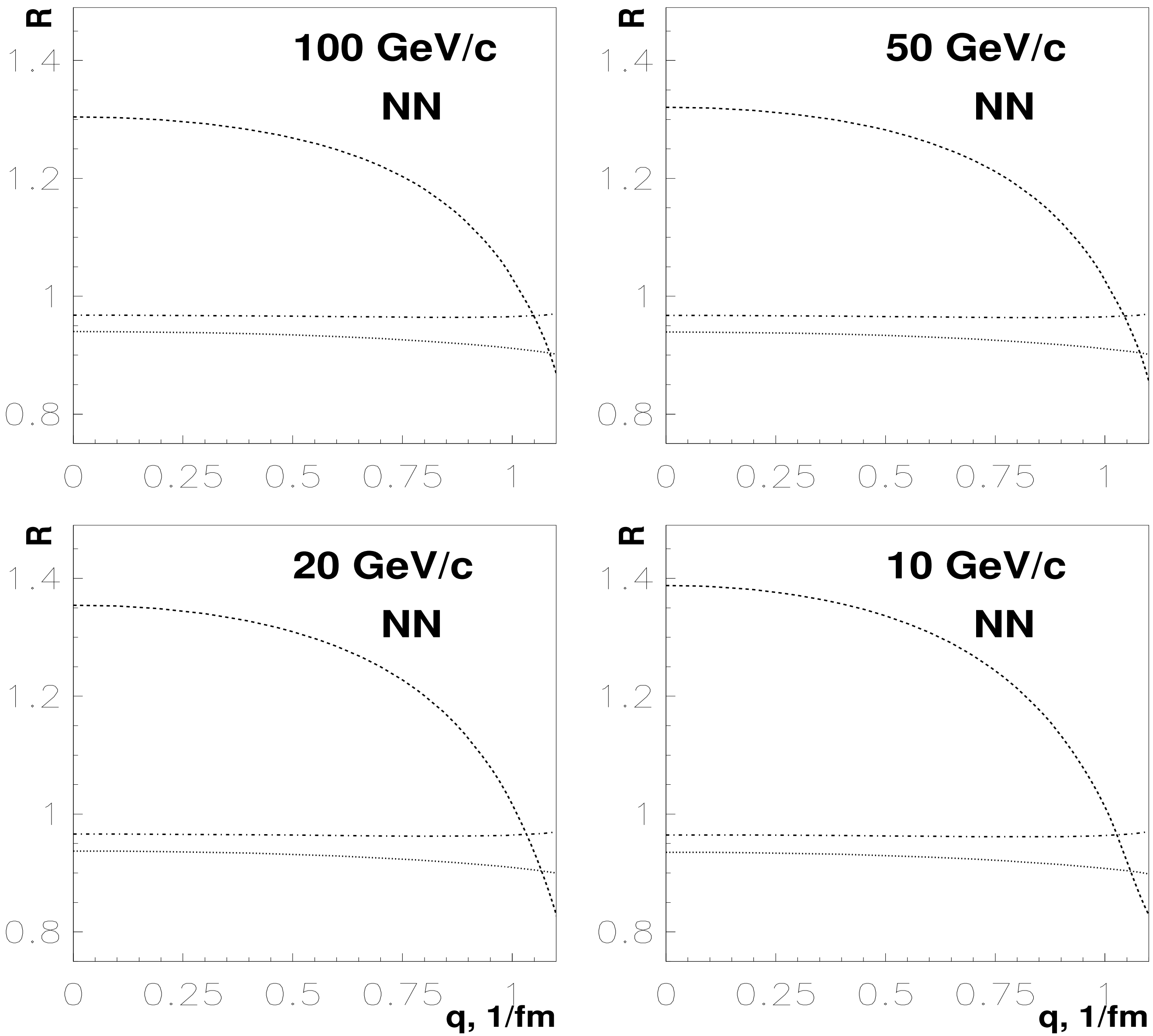

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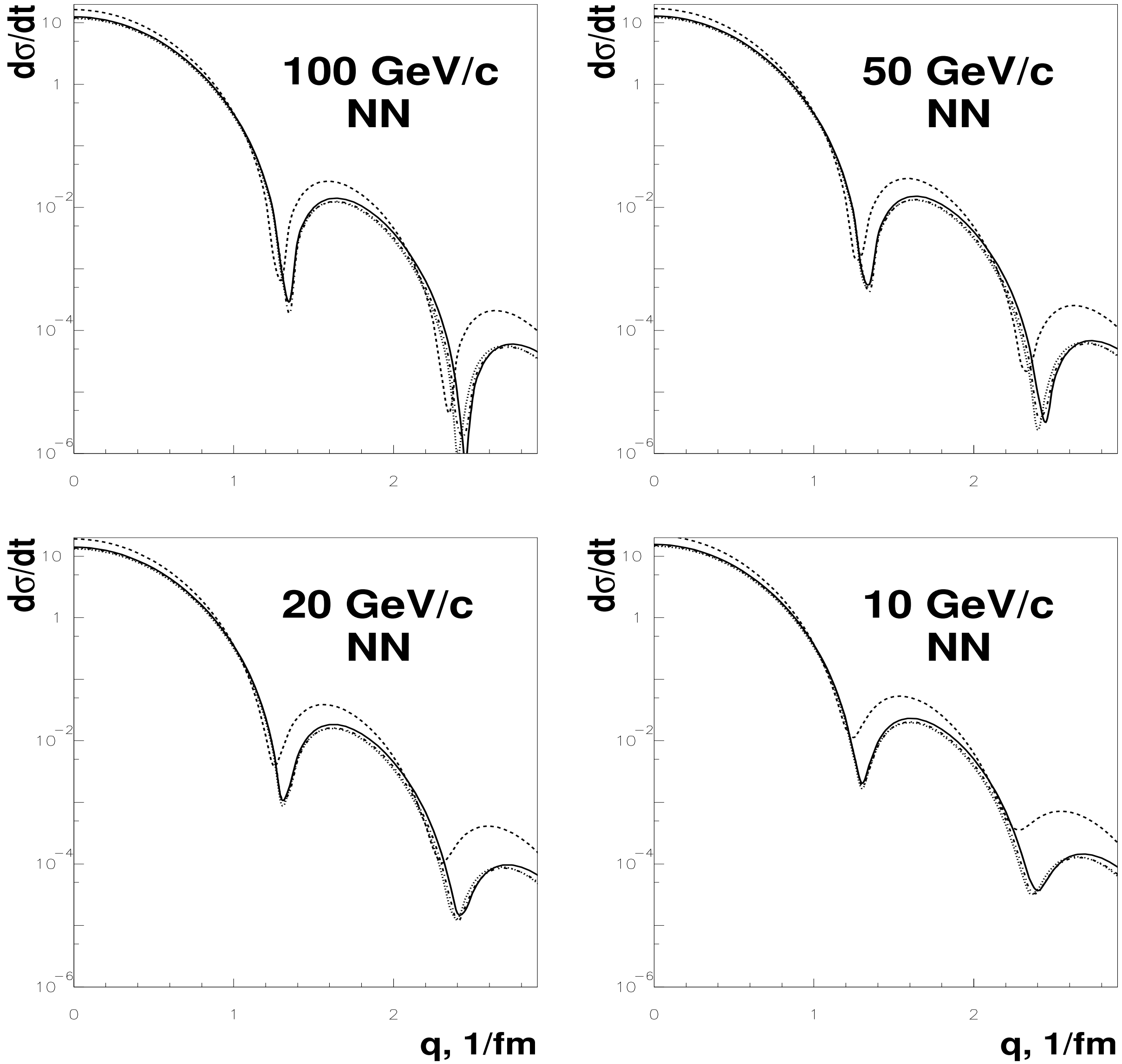

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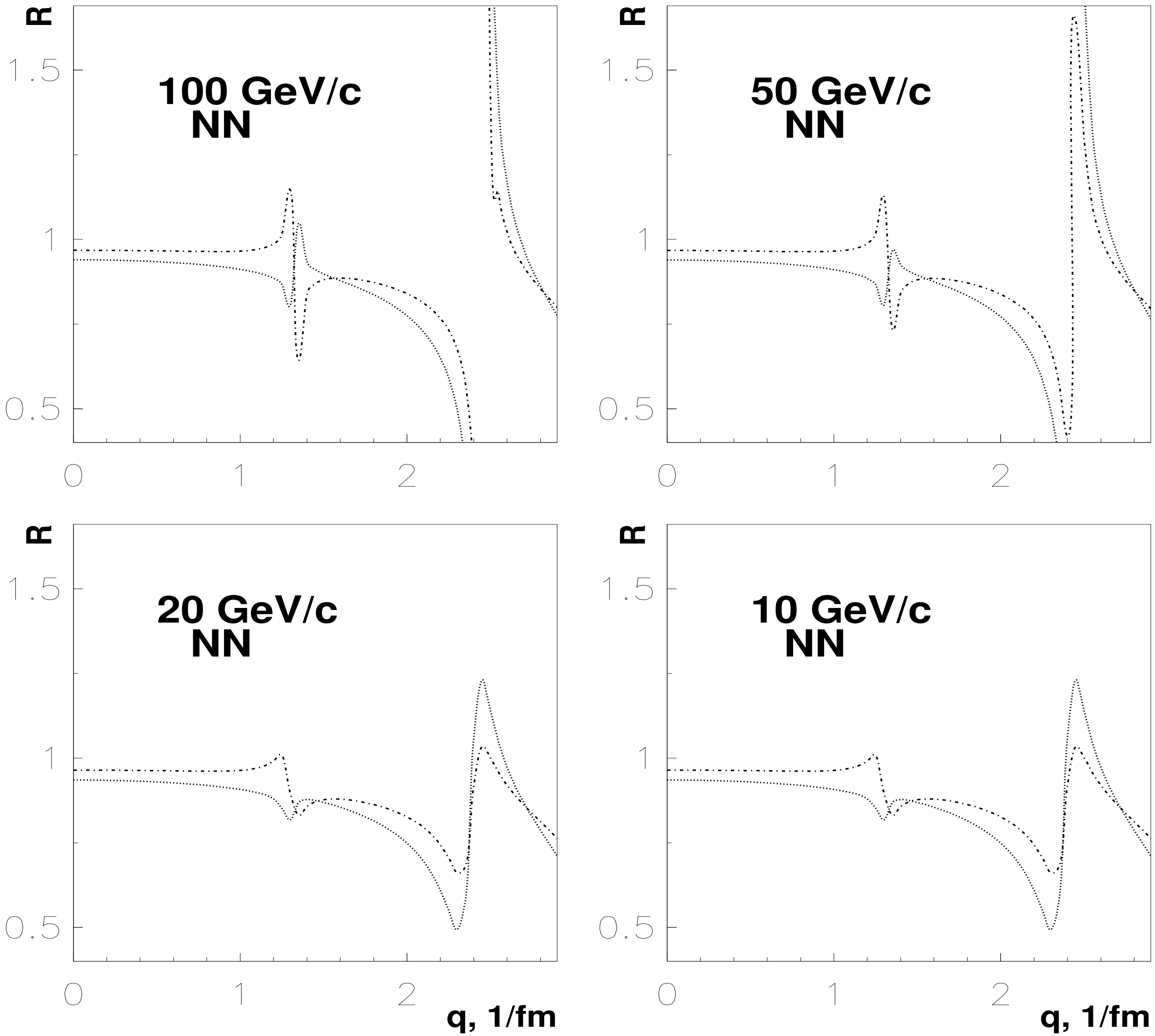


Discussion

- Why the independent particle approximation is the best?
  The profile functions in Glauber theory do not depend on the longitudinal variables z_1, z_2, ..., z_A, hence quantity
  $\int dz_1 dz_2 ... dz_A \langle \Psi_i || \Psi_i \rangle = \tilde{\rho}(\vec{s}_1, \vec{s}_2, ..., \vec{s}_A)$ is considered instead of
  $\langle \Psi_i || \Psi_i \rangle = \rho(\vec{r}_1, \vec{r}_2, ..., \vec{r}_A)$.
  Even for $\vec{s}_1 = \vec{s}_2$ the mean longitudinal distance $|z_1 - z_2|$ in integral over dz_1 and dz_2 is about of the nucleus radius R_A. Hence any correlations (including short-range core correlation) can give small fractional contribution to the integral. We must have a good approximation for transverse densities

  $$\tilde{\rho}(\vec{s}_1, \vec{s}_2, ..., \vec{s}_A) \approx \tilde{\rho}(\vec{s}_1) \cdot \tilde{\rho}(\vec{s}_2) \cdot ... \cdot \tilde{\rho}(\vec{s}_A),$$

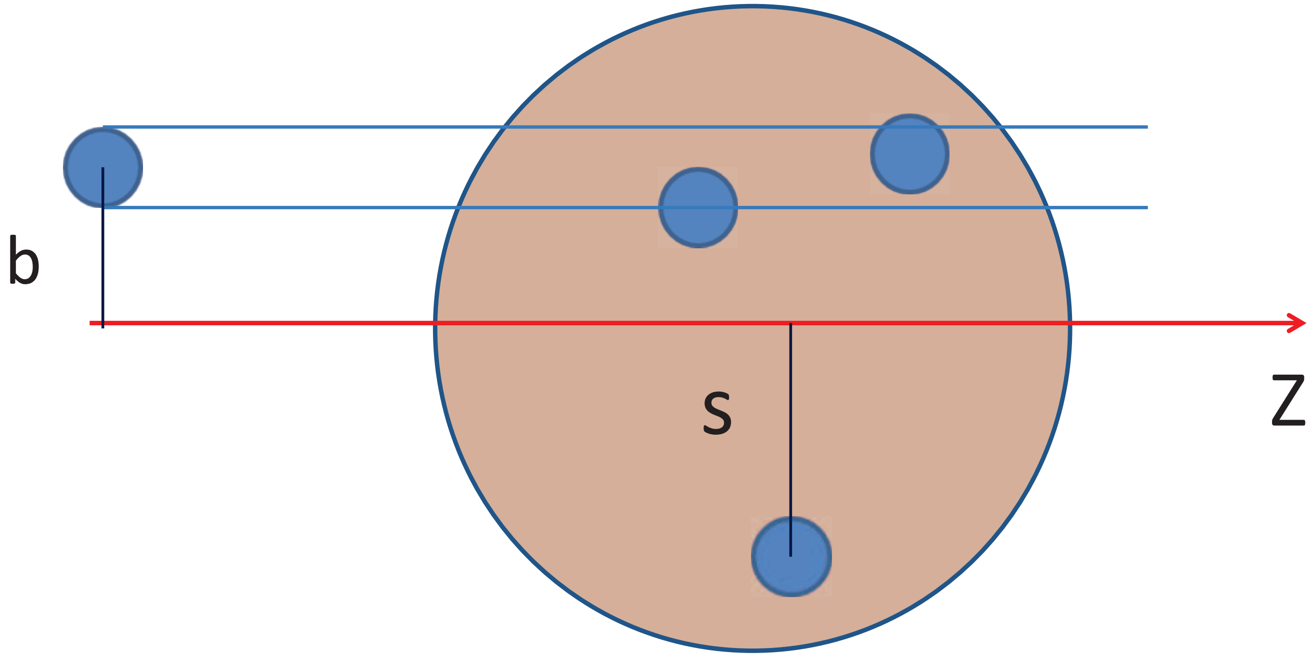
  while the approximation for three-dimensional densities

  $$\rho(\vec{r}_1, \vec{r}_2, ..., \vec{r}_A) \approx \rho(\vec{r}_1) \cdot \rho(\vec{r}_2) \cdot ... \cdot \rho(\vec{r}_A)$$

  is wrong for small relative distances when at least for any j and m $|\vec{r}_j - \vec{r}_m| \ll R_A$.
- The best one-particle approximation for nuclear profile function which provides the best approximation for amplitude of elastic hadron-nucleus scattering is
  $\langle \Psi_i | \Gamma(\vec{b}, \vec{s}_1, \vec{s}_2, ... \vec{s}_A) | \Psi_i \rangle = 1 - [1 - \int \gamma(\vec{b} - \vec{s}) \tilde{\rho}(\vec{s})) d^2\vec{s})]^A$
  which is explained by transverse nature of profile functions in Glauber theory and has nothing to do with completeness relation.

Discussion

- Transverse and longitudinal distances



Summary

- The amplitude of hadron-nucleus scattering is given in Glauber theory in terms of amplitudes of elementary particle scattering and nucleus wave functions. One-particle approximation is needed for medium and heavy nuclei to avoid calculation of many-dimensional integrals.

- The main property of Glauber theory wich permits application of one-particle approximations is dependence of profile functions on two-dimensional vectors orthogonal to the beam direction. This leads to a possibility of using transverse (two-dimensional) nucleon densities for which the approximation of independet particles is valid with rather high precision.

- The best one-particle approximation for nuclear profile function of elastic hadron-nucleus scattering is

$$\langle \Psi_i | \Gamma(\vec{b}, \vec{s}_1, \vec{s}_2, ... \vec{s}_A) | \Psi_i \rangle = 1 - [1 - \int \gamma(\vec{b} - \vec{s}) \tilde{\rho}(\vec{s})) d^2\vec{s})]^A .$$

It provides an accuracy of a few per cents for differential cross section at q of about 1 fm^{-1} even for rather light nuclei as ^{16}O.