Reissner-Nordström solution in tetrad representation as model for classical electron with finite action and total mass

S. I. Manaenkov

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- Gravitation and particle physics
- Reissner-Nordström solution
- Uniform coordinates
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- Equivalence principle for Reissner-Nordström solution
- Conclusions

- Ratio of electrostatic and gravitational forces for two electrons

$$\frac{\mathcal{F}_{el}}{\mathcal{F}_{gr}} = \frac{e^2}{R^2} \Big/ \frac{km^2}{R^2} = \frac{e^2}{km^2} = 4.2 \cdot 10^{42},; \frac{e}{\sqrt{km}} = 2.05 \cdot 10^{21}$$

where $e = 4.8 \cdot 10^{-10}$ CGSE, $m = 9.11 \cdot 10^{-28}$ g, $k = 6.67 \cdot 10^{-8}$ cm^3 g^{-1} c^{-2}.

- The gravitation field does not play any role in the elementary particle structure!
  Nevertheless this statement could be wrong.
- Infinite electromagnetic mass of the electron

$$m_{em} = \frac{1}{c^2}\int \frac{\vec{E}^2}{8\pi} dV = \frac{1}{c^2}\int \frac{e^2}{8\pi R^4} 4\pi R^2 dR,$$

$$m_{em}c^2 = \int_{r_{cl}}^{\infty} \frac{e^2 dR}{2R^2} = e^2/(2r_{cl}) = mc^2,$$

where $r_{cl} = 1.4 \cdot 10^{-13}$ cm is the classical electron radius.

- Gravitational interaction in Newtonian physics

$$dU_{gr} = -\frac{k dm_1 dm_2}{R_{12}} = -\frac{e^4 k}{(8\pi)^2 c^4 R_{12}} \frac{dV_1}{R_1^4} \frac{dV_2}{R_2^4} < 0.$$

$U_{gr}/\mathcal{E}_{em} \sim -r_e^2/R_c^2$, where $r_e^2 = ke^2/c^4$, $r_e = 1.4 \cdot 10^{-34}$ cm.

Gravitation can play important role in classic physics at $r \sim r_e$.

- Electromagnetic and gravitational field equations

  $F^{ik}_{;k} = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^k}\left\{\sqrt{-g}F^{ik}\right\} = 0$ where $g = \det[g_{ik}]$.

  $R^i_k - \frac{1}{2}\delta^i_k R = \frac{8\pi k}{c^4}T^i_k, \quad T^i_k = \frac{1}{4\pi}\left\{-F^{il}F_{kl} + \frac{1}{4}\delta^i_k F_{lm}F^{lm}\right\}$.

- Reissner-Nordström solution

  Spherical coordinates: $x^0 = ct$, $x^1 = r$, $x^2 = \theta$, $x^3 = \varphi$.

  $F^{10} = -F^{01} = F_{10} = -F_{01} = E_r = \frac{e}{r^2}$,

  $T^0_{(em)0} = T^1_{(em)1} = -T^2_{(em)2} = -T^3_{(em)3} = \frac{e^2}{8\pi r^4}, \quad T^j_{(em)j} = 0$.

  $g_{00} = 1/g^{00} = \Lambda \equiv 1 - \frac{2km}{c^2 r} + \frac{ke^2}{c^4 r^2} = 1 - \frac{r_g}{r} + \frac{(r_e)^2}{r^2}$,

  $r_g = \frac{2km}{c^2} = 1.35 \cdot 10^{-55}\text{cm}$, $R_{Pl} = 1.6 \cdot 10^{-33}\text{cm}$,

  $r_e^2 = \frac{ke^2}{c^4}$, $r_e = 1.38 \cdot 10^{-34}\text{cm}$, $r_e \gg r_g$.

  $g_{rr} \equiv g_{11} = 1/g^{11} = -1/\Lambda = -\left[1 - \frac{2km}{c^2 r} + \frac{ke^2}{c^4 r^2}\right]^{-1}$,

  $g_{\theta\theta} \equiv g_{22} = 1/g^{22} = -r^2$, $g_{\varphi\varphi} = g_{33} = 1/g^{33} = -r^2\sin^2\theta$.

- Relation between radii r and ρ ($r \geq 0$, $\rho \geq \rho_{min}$)

$$r = \rho \mathcal{D}(\rho),$$

$$\mathcal{D}(\rho) = 1 + \frac{r_g}{2\rho} - \frac{r_0^2}{4\rho^2} \equiv \left[1 + \frac{r_g}{4\rho}\right]^2 - \frac{r_e^2}{4\rho^2},$$

$$\mathcal{N}(\rho) \equiv \frac{dr}{d\rho} = 1 + \frac{r_0^2}{4\rho^2} > 0, \quad r_0^2 = r_e^2 - r_g^2/4.$$

- Uniform coordinates $\rho^0 = x^0$, $\rho^1 = \rho_x$, $\rho^2 = \rho_y$, $\rho^3 = \rho_z$

$$\rho_x = \rho \sin\theta \cos\varphi, \quad \rho_y = \rho \sin\theta \sin\varphi, \quad \rho_z = \rho \cos\theta.$$

- Spacetime interval for Reissner-Nordström (RN) solution

$$ds^2 = g_{ik} d\rho^i d\rho^k \; g_{00} = \frac{\mathcal{N}^2}{\mathcal{D}^2}, \; g_{\mu\mu} = -\mathcal{D}^2, \mu = x, y, z.$$

$$= \frac{\mathcal{N}^2}{\mathcal{D}^2}(d\rho^0)^2 - \mathcal{D}^2(d\rho_x^2 + d\rho_y^2 + d\rho_z^2).$$

$$\mathcal{D}(\rho) = 0, \quad \rho = \rho_{min} = r_e/2 - r_g/4.$$

- Definition of tetrads and their properties

  Basis of four unit mutually orthogonal four-vectors $h^i_{(a)}$ defined in every point of spacetime;

  $a = 0,\ 1,\ 2,\ 3$ is the vector number,
  $i = 0,\ 1,\ 2,\ 3$ denotes its contravariant spacetime component:
  $h^0_{(a)} = h_{(a)t},\ h^1_{(a)} = h_{(a)x},\ h^2_{(a)} = h_{(a)y},\ h^3_{(a)} = h_{(a)z}.$

  $h_{(a)i} = g_{ik} h^k_{(a)}, \quad h^i_{(a)} = g^{ik} h_{(a)k}, \quad h_{(a)i} h^i_{(b)} = \eta_{ab}.$

  $\mathsf{h}^{(a)i} = \eta^{ab} h^i_{(b)},\ h^i_{(a)} = \eta_{ab} h^{(b)i},$

  $\eta^{ab} = \eta_{ab} = \mathrm{diag}(1, -1, -1, -1).$

- Main properties

$$h_{(a)i} h^{(a)}_k = g_{ik}, \quad h^i_{(a)} h^{(a)k} = g^{ik}.$$

$$g \equiv \det[\mathrm{g_{ik}}] = -|\mathrm{h}|^2, \quad |\mathrm{h}| = \det[\mathrm{h_{(a)i}}].$$

- Tetrads for Reissner-Nordström solution

  Nonzero components of tetrad four-vectors $h_i^{(a)}$ are:

  $h_{(0)0} = \frac{\mathcal{N}}{\mathcal{D}},\ h_{(1)x} = h_{(2)y} = h_{(3)z} = \mathcal{D},$

  $h_0^{(0)} = \frac{\mathcal{N}}{\mathcal{D}},\ h_x^{(1)} = h_y^{(2)} = h_z^{(3)} = -\mathcal{D}.$

  $g_{00} = h_{(a)0} h_0^{(a)} = h_{(0)0} h_0^{(0)} = h_{(0)0} h_{(0)0} = \frac{\mathcal{N}^2}{\mathcal{D}^2},$

  $g_{xx} = h_{(a)x} h_x^{(a)} = h_{(1)x} h_x^{(1)} = -h_{(1)x} h_{(1)x} = -\mathcal{D}^2,$

  $g_{yy} = h_{(a)y} h_y^{(a)} = h_{(2)y} h_y^{(2)} = -h_{(2)y} h_{(2)y} = -\mathcal{D}^2,$

  $g_{33} = h_{(a)3} h_3^{(a)} = h_{(3)3} h_3^{(3)} = -h_{(3)z} h_{(3)z} = -\mathcal{D}^2,$

  $g_{ik} = 0$ if $i \neq k$, $|h| \equiv \det[\mathrm{h_{(a)i}}] = \mathcal{N}\mathcal{D}^2.$

- RN tetrads obey Euler-Lagrange equation

  $\frac{\delta \mathcal{L}_{tot}}{\delta h_p^{(c)}} = \frac{\partial}{\partial \rho^q} \left\{ \frac{\delta \mathcal{L}_{tot}}{\delta h_{p,q}^{(c)}} \right\},$ where $h_{p,q}^{(c)} = \frac{\partial h_p^{(c)}}{\partial \rho^q}.$

- Lagrangian density for gravitation field
  Einstein's theory: $\mathcal{L}_g = -\frac{R\sqrt{-g}}{2\kappa}, \quad \kappa = \frac{8\pi k}{c^4}$.
  Since $R = -\kappa T^s_{(em)s} = 0$ the Hilbert-Einstein action $S_g = \frac{1}{c}\int \mathcal{L}_g d^4x = 0$ if there are gravitational and electromagnetic fields only.
- Lagrangian density of electromagnetic field
  Electromagnetic field action
  $S_{em} = \frac{1}{c}\int \mathcal{L}_{em} d^4x = -\frac{1}{16\pi c}\int F_{ik}F^{ik}\sqrt{-g}d^4x$.
  For Reissner-Nordström solution,
  $S_{em} = \frac{1}{8\pi}\int \vec{E}^2 dV dt = \infty$.
  The total action $S_{tot} = S_g + S_{em}$ is meaningless for RN solution in Einstein's theory.
  For RN solution $\mathcal{L}'_g = \frac{\sqrt{-g}g^{ik}}{2\kappa}[\Gamma^s_{ir}\Gamma^r_{ks} - \Gamma^r_{ik}\Gamma^s_{rs}] = 0$ also.

Action for gravitational and electromagnetic fields

- Lagrangian density $\mathcal{L}_g$ in tetrad representation

  Moller's formula: $\mathcal{L}_g = \frac{|h|}{2\kappa}\Big(h^k_{(a);l}h^{(a)l}{}_{;k} - h^k_{(a);k}h^{(a)l}{}_{;l}\Big).$

- Total Lagrangian density $\mathcal{L}_{tot} = \mathcal{L}_g + \mathcal{L}_{em}$ for RN solution

  $\mathcal{L}_{tot} = \frac{r_0^2}{\kappa\rho^4}, \quad r_0^2 = r_e^2 - r_g^2/4, \quad r_e^2 = \frac{ke^2}{c^4}, \quad r_g = \frac{2km}{c^2}, \quad \kappa = \frac{8\pi k}{c^4}.$

- Total Lagrangian and action for RN solution

  $\mathsf{L}_{tot} = \int_{\rho\geq\rho_{min}} \mathcal{L}_{tot} d^3\rho = 4\pi\frac{r_0^2}{\kappa\rho_{min}}, \quad \rho_{min} = r_e/2 - r_g/4.$

  $L_{tot} = \frac{ec^2}{\sqrt{k}} + mc^2, \quad S_{tot} = \Big(\frac{ec^2}{\sqrt{k}} + mc^2\Big)t.$

  Total Lagrangian and action are finite for RN solution in tetrad representation in spite of singularities of electromagnetic and gravitational fields at $\rho = \rho_{min}$ $(r = 0)$.

- Asymptotic behaviour of g_{00} at $\rho \to \infty$ $(r \to \infty)$

$$g_{00} \approx 1 + 2\frac{\phi(\rho)}{c^2} = 1 - \frac{2km_{gr}}{c^2\rho},$$

where $\phi(\rho)$ is Newtonian potential.

For RN solution $g_{00} = \frac{\mathcal{N}^2}{\mathcal{D}^2} \approx 1 - \frac{r_g}{\rho} = 1 - \frac{2km}{c^2\rho}$, $m = m_{gr}$.

- Total energy-momentum pseudo-tensor and superpotential

$$T_i^k = \frac{\partial U_i{}^{kl}}{\partial \rho^l},\ P_i = \frac{1}{c}\int_V T_i^0 d^3\rho = \frac{1}{c}\int_\Sigma U_i{}^{0\lambda} k_\lambda d\sigma.$$

$$U_i{}^{kl} = \frac{|h|}{\kappa}\left\{h^k_{(a)} h^{(a)l}{}_{;i} + \left(\delta_i^k h^{(a)l} - \delta_i^l h^{(a)k}\right) h^s_{(a);s}\right\}.$$

For Reissner-Nordström solution $U_0{}^{0\lambda} = -2\left(\frac{\rho^\lambda}{\kappa\rho}\right)\frac{\mathcal{N}(\rho)\mathcal{D}'(\rho)}{\mathcal{D}(\rho)}$.

Superpotential $U_0{}^{0\lambda}$ is infinite at $\rho = \rho_{min}$ since $\mathcal{D}(\rho) = 0$, therefore expression for energy becomes meaningless.

Equivalence principle for Reissner-Nordström solution

- Superpotential ${W_i}^{kl}(\rho)$
  ${W_i}^{kl} = |h|^n {U_i}^{kl}$ for $n \geq 1$. The case $n = 1$ is used.

  For RN solution, ${W_0}^{0\lambda} = -2\left(\frac{\rho^\lambda}{\kappa\rho}\right)\mathcal{N}^2\mathcal{D}'\mathcal{D}$, ${W_\mu}^{0\lambda} = 0$.

  Since $|h|(\rho) = \mathcal{N}\mathcal{D}^2 = 0$ when $\rho = \rho_{min}$,
  then ${W_i}^{0\lambda}(\rho) \to 0$ at $\rho \to \rho_{min}$.
  If $|h|(\rho) \to |h|_\infty$ when $\rho \to \infty$, therefore
  ${W_i}^{0\lambda}(\rho) \to |h|_\infty {U_i}^{0\lambda}(\rho)$
  $\tilde{P}_i = \frac{1}{c}\int_\Sigma {W_i}^{0\lambda} k_\lambda d\sigma = \frac{|h|_\infty}{c}\lim_{R\to\infty}\left\{\int_{\Omega_R} {U_i}^{0\lambda}\left(\frac{\rho_\lambda}{\rho}\right)d\sigma\right\}.$

  $\tilde{P}_i = P_i |h|_\infty$. For RN solution $|h|_\infty = 1$, $\tilde{P}_i = mc\delta_i^0$,

  hence $m_{inert} = m = m_{gr}$ (equivalence principle).
- It is possible to use the definition of the energy-momentum vector which does not loose its meaning for solutions with singular electromagnetic and gravitational fields.

Conclusions

- Classical electron is a system of electromagnetic and gravitational fields localized in a space region of a range $r_e \sim 10^{-34}$ cm. It is described with the Reissner-Nordström (RN) solution in the tetrad representation with parameters e and m equal to the experimental electron electrical charge and mass, respectively.
- The total Lagrangian density of this system and action are finite for the tetrad representation.
- The superpotential can be defined in such a way that it provides the finite total inert mass for the RN solution.
- The equivalence principle for the classical electron under discussion is fulfilled.
- There is no need in additional point-like particles having the charge e and any bare mass.
- Charge conjugation ($e \to -e$, $m \to m$, $\vec{E} \to -\vec{E}$, $g_{ij} \to g_{ij}$) of the solution for the electron gives the solution for the positron.
  It has the same positive mass as the electron.